

MIXED n -STEP MIR INEQUALITIES, n -STEP CONIC MIR INEQUALITIES
AND
A POLYHEDRAL STUDY OF SINGLE ROW FACILITY LAYOUT PROBLEM

A Dissertation
by
SUJEEVRAJA SANJEEVI

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

August 2012

Major Subject: Industrial Engineering

MIXED n -STEP MIR INEQUALITIES, n -STEP CONIC MIR INEQUALITIES
AND
A POLYHEDRAL STUDY OF SINGLE ROW FACILITY LAYOUT PROBLEM

A Dissertation
by
SUJEEVRAJA SANJEEVI

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Kiavash Kianfar
Committee Members,	Lewis Ntaimo
	Wilbert Wilhelm
	Donald Friesen
Head of Department,	Cesar Malave

August 2012

Major Subject: Industrial Engineering

ABSTRACT

Mixed n -Step MIR Inequalities, n -Step Conic MIR Inequalities

and

A Polyhedral Study of Single Row Facility Layout Problem. (August 2012)

Sujeevraja Sanjeevi, B.E, Anna University; M.S., Texas A&M University

Chair of Advisory Committee: Dr. Kiavash Kianfar

In this dissertation, we introduce new families of valid inequalities for general linear mixed integer programs (MIPs) and second-order conic MIPs (SOCMIPs) and establish several theoretical properties and computational effectiveness of these inequalities.

First we introduce the mixed n -step mixed integer rounding (MIR) inequalities for a generalization of the mixing set which we refer to as the n -mixing set. The n -mixing set is a multi-constraint mixed integer set in which each constraint has n integer variables and a single continuous variable. We then show that mixed n -step MIR can generate multi-row valid inequalities for general MIPs and special structure MIPs, namely, multi-module capacitated lot-sizing and facility location problems. We also present the results of our computational experiments with the mixed n -step MIR inequalities on small MIPLIB instances and randomly generated multi-module lot-sizing instances which show that these inequalities are quite effective.

Next, we introduce the n -step conic MIR inequalities for the so-called polyhedral second-order conic (PSOC) mixed integer sets. PSOC sets arise in the polyhedral reformulation of SOCMIPs. We first introduce the n -step conic MIR inequality for a PSOC set with n integer variables and prove that all the 1-step to n -step conic MIR inequalities are facet-defining for the convex hull of this set. We also provide necessary and sufficient conditions for the PSOC form of this inequality to be valid. Then, we use

the aforementioned n -step conic MIR facet to derive the n -step conic MIR inequality for a general PSOC set and provide conditions for it to be facet-defining. We further show that the n -step conic MIR inequality for a general PSOC set strictly dominates the n -step MIR inequalities written for the two linear constraints that define the PSOC set. We also prove that the n -step MIR inequality for a linear mixed integer constraint is a special case of the n -step conic MIR inequality.

Finally, we conduct a polyhedral study of the triplet formulation for the single row facility layout problem (SRFLP). For any number of departments n , we prove that the dimension of the triplet polytope (convex hull of solutions to the triplet formulation) is $n(n-1)(n-2)/3$. We then prove that several valid inequalities presented in Amaral (2009) for this polytope are facet-defining. These results provide theoretical support for the fact that the linear program solved over these valid inequalities gives the optimal solution for all instances studied by Amaral (2009).

To Sivaramakrishnan Srinivasan

ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor Dr. Kiavash Kianfar for introducing me to the joy of research in mixed integer programming. Without his guidance, support, strictness, joviality, excellence in teaching and emphasis on intuition, none of this would have been possible. I would like to thank my committee members Dr. Lewis Ntaimo, Dr. Wilbert Wilhelm and Dr. Donald Friesen for their valuable inputs. I would also like to thank Dr. Martin Wortman and Dr. Justin Yates for introducing me to research during my masters program.

I am indebted and eternally grateful to thank my parents and my brother for their patience and support throughout my period of education. I am also grateful to my late uncle Mr. Sivaramakrishnan Srinivasan, who introduced me to the wonder that is mathematics. He led me along a path of discovery of fundamental mathematical concepts through a series of puzzles and discussions over a period of several years, and instilled in my mind a permanent love for the subject.

TABLE OF CONTENTS

CHAPTER		Page
I	INTRODUCTION AND SUMMARY OF CONTRIBUTIONS	1
	I.1 Mixed n -step MIR Inequalities	2
	I.2 n -step Conic MIR Inequalities	3
	I.3 Polyhedral Study of the Triplet Formulation for SRFLP .	4
	I.4 Dissertation Structure	5
II	MIXED INTEGER PROGRAMMING AND MIXED INTEGER ROUNDING	6
	II.1 Mixed Integer Programming	6
	II.1.1 Polyhedral Definitions	8
	II.1.2 Solution Algorithms for MIPS	9
	II.2 Mixed Integer Rounding	14
	II.3 n -step MIR Inequalities	16
	II.4 Mixing Inequalities	19
	II.4.1 Valid Inequalities for Production Planning Problems	21
	II.4.2 Valid Inequalities for Capacitated Facility Loca- tion (CFL) Problems	22
	II.5 Conic MIR Inequalities	24
III	MIXED n -STEP MIR INEQUALITIES	28
	III.1 Mixed n -step MIR Inequalities for the n -mixing Set . . .	30
	III.2 Facets Defined by Mixed n -step MIR Inequalities	34
	III.3 Mixed n -step MIR Inequalities for General MIP	45
	III.4 Mixed n -step MIR Inequalities for Special Structures . .	50
	III.4.1 Multi-Module Lot-Sizing (MML)	50
	III.4.2 Multi-Module Facility Location (MMF)	55
	III.5 Computational Results	58
	III.5.1 MIPLIB Instances	58
	III.5.2 Multi-module Lot-sizing Instances	62
	III.6 Concluding Remarks	64
IV	n -STEP CONIC MIR INEQUALITIES	66
	IV.1 Preliminaries	67
	IV.2 n -step Conic MIR Facet for Q^n	71

IV.3	Polyhedral Second-order Conic Inequalities	78
IV.4	n -step Conic MIR Inequality for the General PSOC Set S	84
IV.5	n -step Conic MIR dominates n -step MIR for S	92
IV.6	Concluding Remarks	95
V	A POLYHEDRAL STUDY OF TRIPLET FORMULATION FOR SINGLE ROW FACILITY LAYOUT PROBLEM	97
V.1	Triplet polytope, its projections and valid inequalities . .	99
V.2	Dimension of convex hulls of P , P^1 and P^2	101
V.3	Facet-defining properties of valid inequalities	105
V.4	Conclusions	112
VI	CONCLUSION AND FUTURE RESEARCH	113
VI.1	Theoretical Research	114
VI.2	Computational Research	116
	REFERENCES	117

LIST OF TABLES

TABLE		Page
I	Results of computational experiments on small MIPLIB instances	62
II	Results of computational experiments on MML instances	64

LIST OF FIGURES

FIGURE		Page
1	$\sigma_{\alpha,b}^2(d_1, d_2)$ over $[-25, 25]^2$ with $\alpha = (25, 10)$ and $b = (39, 18)$	49
2	$\phi_{\alpha,0.8}^1(u), \alpha = 1$	88
3	$\phi_{\alpha,0.8}^2(u), \alpha = (1, 0.3)$	88
4	$\phi_{\alpha,0.8}^3(u), \alpha = (1, 0.3, 0.08)$	89
5	$\phi_{\alpha,0.8}^4(u), \alpha = (1, 0.3, 0.08, 0.025)$	89

CHAPTER I

INTRODUCTION AND SUMMARY OF CONTRIBUTIONS

Mixed Integer Programs (MIP) are powerful optimization tools with several applications in business, engineering and science. MIPs are NP-hard problems in general, and several solution approaches have been proposed to solve MIPs. One of the primarily used solution techniques is the branch-and-cut algorithm. Valid inequalities or cutting planes that tighten the continuous relaxation of MIPs in order to achieve a better approximation of the convex hull of feasible solutions are an integral part of branch-and-cut algorithms. Development of cutting planes is a research direction that has been actively pursued in the last few decades.

The research in this dissertation focuses on developing new classes of strong valid inequalities for linear and second-order conic mixed integer programs (MIPs and SOCMIPs), and establishing several theoretical properties of these valid inequalities. The intellectual contributions of this research are threefold:

- Develop *mixed n -step MIR inequalities* for general and special structure linear MIPs, and establish several theoretical properties and the computational effectiveness of these valid inequalities.
- Develop *n -step conic MIR inequalities* for SOCMIPs and linear MIPs, and establish the theoretical properties of these valid inequalities.
- Conduct a polyhedral study of a MIP formulation referred to as the *triplet formulation for the single row facility layout problem (SRFLP)* and provide theoretical

The journal model is Mathematical Programming.

support for the successful computational results obtained using this formulation.

Our research results have two major *impacts* on the field of integer optimization: (1) Cutting planes are a crucial part of almost all algorithms used to solve MIPs. The new classes of valid inequalities developed in this research can be used as new cutting planes in solving general and special structure MIPs and SOCMIPs resulting in potentially faster algorithms. (2) This research adds to the theoretical knowledge on valid inequalities for MIPs and SOCMIPs by generalizing some well-known special mixed-integer sets, proving several theoretical properties for each set, and using these properties to study more general multi-constraint mixed-integer sets. The generalized view resulting from the proposed research provides valuable insight into the polyhedral structure of the aforementioned sets. This insight opens doors to several new lines of research in this area. In the following sections, we present a brief summary of our research contributions and the organization of the remainder of this dissertation.

I.1 Mixed n -step MIR Inequalities

Understanding the polyhedral structure of simple mixed-integer sets and using it to develop valid inequalities for general MIPs has been a successful approach. One such simple set is the mixing set introduced by Günlük and Pochet [70]. This set has multiple linear constraints each containing a single integer variable. Using a procedure called mixing of mixed integer rounding (MIR) inequalities, Günlük and Pochet [70] developed facet-defining valid inequalities, called mixed MIR inequalities, for the mixing set. They then utilized these inequalities to generate valid inequalities for multi-constraint general and special-structure MIPs.

In this dissertation, we introduce the *mixed n -step MIR inequalities* through a generalization of the mixed MIR inequalities of Günlük and Pochet [70]. The mixed MIR

inequalities are simply the special case of $n = 1$. We show that mixed n -step MIR inequalities define facets and high-dimensional faces for the convex hull of a generalization of the mixing set where each constraint contains multiple integer variables. We refer to this set as the n -mixing set. We then use the mixed n -step MIR inequalities to develop new valid inequalities for general MIPs as well as special-structure MIPs, namely multi-module lot-sizing and multi-capacity facility location problems. The valid inequalities developed in [70] for general MIPs and the single-capacity lot-sizing and facility location problems are special cases of our inequalities. We also present the results of our computational experiments conducted to test the effectiveness of the mixed n -step MIR cuts for general MIPs and multi-module lot sizing problems.

I.2 n -step Conic MIR Inequalities

A second-order conic mixed-integer program (SOCMIP) is a second-order cone programming (SOCP) problem in which at least one variable is required to be integer. Linear programming, quadratically constrained quadratic programming and several more general convex optimization problems can be formulated as SOCP [84]. Hence, by adding integrality requirement to a subset of variables in any of these problems, they can be formulated as an SOCMIP. Some important applications of SOCMIP are in portfolio optimization [28, 27, 84, 85, 91] and signal processing [87, 88, 95]. The conic constraint of a SOCMIP has a polyhedral reformulation in a higher-dimensional space [19]. This reformulation has constraints in which the left-hand side is the absolute value of a linear function of variables and the right-hand side is a continuous variable. These constraints are referred to as the polyhedral second-order conic (PSOC) constraints. Valid inequalities developed for PSOC sets can be added to the original SOCMIP as cutting planes. Atamtürk and Narayanan [19] developed a facet for a PSOC set with

a single integer variable and used it to develop the so-called conic MIR inequality for a general PSOC set.

In this dissertation, we introduce a new facet for a PSOC set with n integer variables [93]. The simple conic MIR inequality of Atamtürk and Narayanan is a special case of this facet. We then use n -step conic MIR faces for lower dimensional sets to generate facets for higher dimensional PSOC sets. The n -step conic MIR facets are linear inequalities. We use them to generate nonlinear valid inequalities for the original SOCMIP. We also develop new valid inequalities for general PSOC sets using the n -step conic MIR facet, and identify conditions under which they are facet-defining. Finally, we use the n -step conic MIR facets to develop new two-row valid inequalities for linear MIPs. We also show that the n -step MIR inequalities of Kianfar and Fathi [78] can be generated using n -step conic MIR.

I.3 Polyhedral Study of the Triplet Formulation for SRFLP

In a different direction, we proved that several valid inequalities proposed for the triplet formulation of the SRFLP by Amaral [9] are facet-defining. SRFLP is the problem of arranging n departments with given lengths on a straight line so as to minimize the total weighted distance between all department pairs. The *Minimum Linear Arrangement Problem (MLAP)* was proven to be NP-hard in [60]. The SRFLP is a generalization of MLAP and so is also NP-hard. Amaral [9] presented a MIP formulation for the SRFLP, here referred to as the *triplet formulation*, and introduced a set of valid inequalities for it. Surprisingly, the linear program solved over these valid inequalities yields the optimal solution for several classical SRFLP instances of sizes $n = 5$ to 30 [9].

In this dissertation, we first prove that the triplet polytope for n departments is of dimension $n(n-1)(n-2)/3$. Then we prove that almost all valid inequalities introduced

in [9] are facet-defining [106] providing theoretical support for the computational results in [9]. We also show that similar results hold for the other two projections of the triplet polytope introduced in [9].

I.4 Dissertation Structure

The dissertation is organized as follows: after a brief review of mixed-integer programming, and relevant definitions and results required to present our research in Chapter II, we present our research on mixed n -step MIR, n -step conic MIR and SRFLP in Chapters III, IV and V respectively. Finally, we conclude in Chapter VI with a brief discussion on future research plans beyond this dissertation.

CHAPTER II

MIXED INTEGER PROGRAMMING AND MIXED INTEGER ROUNDING

In this chapter, we present an introduction to mixed integer programming and review some related concepts required to present our research results. We begin with a discussion on the importance and applications of mixed integer programming, basic polyhedral definitions and solution algorithms for mixed integer programming problems in Section II.1. We also discuss the different types of cutting planes in this section. Next, we present a brief introduction to mixed integer rounding, an approach used to generate cutting planes for mixed integer programs in Section II.2. We then briefly review the different generalizations of mixed integer rounding, namely the n -step mixed integer rounding inequalities, mixing inequalities and conic mixed integer rounding inequalities in Sections II.3, II.4 and II.5 respectively.

II.1 Mixed Integer Programming

Mixed Integer Programming is a powerful and flexible optimization paradigm with ubiquitous applications in business, engineering, and science [97, 117]. Operations and crew scheduling, production and electricity generation planning, facility location, telecommunication and transportation, cutting stock problems, network design and optimization problems are examples of the wide range of MIP applications [117]. Yet solving MIPs is NP-hard in general, and therefore, finding more efficient algorithms for this purpose is a challenging task with substantial impact.

A mixed integer program (MIP) can be formulated as

$$\begin{aligned}
 & \min \quad cx + dy \\
 & \text{s.t.} \quad Ax + Gy \geq b \\
 & \quad x \in \mathbb{Z}^n, y \in \mathbb{R}^p.
 \end{aligned} \tag{MIP}$$

where x, y are the decision variables, and c, d, A, G, b are vectors and matrices of appropriate dimension, assumed to contain rational data. Two special cases of MIP are the pure integer program, which contains only integer variables

$$\begin{aligned}
 & \min \quad cx \\
 & \text{s.t.} \quad Ax \geq b \\
 & \quad x \in \mathbb{Z}^n
 \end{aligned} \tag{IP}$$

and the binary integer program, which contains only binary variables:

$$\begin{aligned}
 & \min \quad cx \\
 & \text{s.t.} \quad Ax \geq b \\
 & \quad x \in \{0, 1\}^n.
 \end{aligned} \tag{IP}$$

The linear relaxation of a MIP is a linear programming problem is obtained by dropping the integrality restrictions on decision variables.

$$\begin{aligned}
 & \min \quad cx + dy \\
 & \text{s.t.} \quad Ax + Gy \geq b \\
 & \quad (x, y) \in \mathbb{R}^{n+p}.
 \end{aligned} \tag{LP}$$

II.1.1 Polyhedral Definitions

In this section, we reproduce some fundamental definitions and theorems related to mixed integer programming and polyhedra from [97, 117] that will be repeatedly utilized throughout this dissertation.

Definition 1. *The feasible region of a MIP, P_{MIP} is the set of points that satisfy its constraints:*

$$P_{MIP} = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p : Ax + Gy \geq b\}.$$

Definition 2. *A subset of \mathbb{R}^n described by a finite set of linear constraints $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polyhedron.*

Definition 3. *Given a set $X \subseteq \mathbb{R}^n$, the convex hull of X , denoted $\text{conv}(X)$, is defined as: $\text{conv}(X) = \{x : x = \sum_{i=1}^t \lambda_i x^i, \sum_{i=1}^t \lambda_i = 1, \lambda_i \geq 0 \text{ for } i = 1, \dots, t \text{ over all finite subsets } \{x^1, \dots, x^t\} \text{ of } X\}$.*

Theorem 4. *$\text{conv}(P_{MIP})$ is a polyhedron, if the data A, G, b is rational.*

The proof of Theorem 4 is provided in [97].

Definition 5. *An inequality $\pi x \leq \pi_0$ is a valid inequality for $X \subseteq \mathbb{R}^n$ if $\pi x \leq \pi_0$ for all $x \in X$.*

Theorem 6. [97] *If $\pi x \leq \pi_0$ is valid for $X \subseteq \mathbb{R}^n$, it is also valid for $\text{conv}(X)$.*

Definition 7. *The points $x^1, \dots, x^k \in \mathbb{R}^n$ are affinely independent if the $k - 1$ directions $x^2 - x^1, \dots, x^k - x^1$ are linearly independent, or alternatively the k vectors $(x^1, 1), \dots, (x^k, 1) \in \mathbb{R}^{n+1}$ are linearly independent.*

Definition 8. *The dimension of P , denoted $\dim(P)$, is one less than the maximum number of affinely independent points in P .*

- Definition 9.** (i) F defines a face of the polyhedron P if $F = \{x \in P : \pi x = \pi_0\}$ for some valid inequality $\pi x \geq \pi_0$ of P .
- (ii) F is a facet of P if F is a face of P and $\dim(F) = \dim(P) - 1$.
- (iii) If F is a face of P with $F = \{x \in P : \pi x = \pi_0\}$, the valid inequality $\pi x \geq \pi_0$ is said to represent or define the face.

II.1.2 Solution Algorithms for MIPS

In this section, we briefly review three algorithms used to solve MIPs, namely branch and bound, cutting plane algorithm and branch and cut. These three algorithms are the primarily used techniques used by most commercial solvers today. The branch and cut is a *general* algorithm, in the sense that it can solve any MIP and does not utilize underlying problem structure. Several other algorithms can be used to solve MIPs. Some of them include lagrangian dualiry, column generation, semidefinite programming, and heuristics such as tabu search, simulated annealing and genetic algorithms [97, 117].

II.1.2.1 Branch and Bound Algorithm

The branch and bound algorithm was first proposed in [81]. The algorithm works by splitting to the problem into smaller subproblems that can be solved easily, and putting this information back together to solve the original problem. In other words, branch and bound utilizes the following theorem.

Theorem 10. [117] Consider the problem $z = \max\{cx : x \in S\}$. Let $S = S_1 \cup S_2 \cup \dots \cup S_K$ be a decomposition of S into smaller sets, and let $z^k = \max\{cx : x \in S_k\}$ for $k = 1, \dots, K$. Then $z = \max_k z^k$.

The algorithm utilizes a tree structure to solve the MIP. The linear relaxation of the MIP, or any other easily solvable relaxation is solved at the root node. If the solution has integer values for all integer variables, the problem is considered to be solved. Otherwise, child nodes are created for the root node such that the feasible region of each child node is a subset of the feasible region of the root node. One branching rule is to create a child node by adding the constraint $x_i \leq \lfloor x_i^* \rfloor$ to the root node linear relaxation, and another node by adding the constraint $x_i \geq \lceil x_i^* \rceil$ where x_i is an integer variable with the fractional LP solution x_i^* . The reason behind this is that the region $\lfloor x_i^* \rfloor < x_i^* < \lceil x_i^* \rceil$ does not contain any solution with x_i integer, and contains the LP solution with $x_i = x_i^*$. This solution is not considered in both child nodes. Once the child nodes are created, the linear relaxation is solved at each child node. A child node becomes inactive or is *pruned* if one of the following three cases occur:

- *pruned by optimality*: The solution of the linear relaxation at the node has integer values for all integer variables. In this case, the objective value of this node becomes a new upper bound for the objective function of the MIP (if it is lower than the current upper bound).
- *pruned by bound*: The objective value of the linear relaxation at the node is greater than the current upper bound for the MIP objective.
- *pruned by infeasibility*: The linear relaxation is infeasible.

The MIP is solved when all nodes are pruned. More details about this algorithm are available in [97, 117]. In practice, this algorithm may result in a huge number of nodes being created, especially when the ratio of the difference between the MIP and LP objective values to the MIP objective value is quite large.

II.1.2.2 Cutting Plane Algorithm

The cutting plane algorithm was first presented in [62]. This algorithm can either be used to solve an MIP directly, or generate an improved formulation (with a better objective value for the linear relaxation) that can then be solved using a branch and bound algorithm. This algorithm utilizes valid inequalities for the MIP that are violated by the solution to the linear relaxation. Such inequalities can be added to the formulation without affecting the MIP solution, and are called *cutting planes* or *cuts*. Cuts that represent higher dimensional faces of the convex hull of feasible integer solutions to the MIP are better cuts, meaning that they cut off more of the feasible region of the linear relaxation. In this sense, the strongest cuts are those that represent facets of the convex hull of integer solutions to the MIP.

The cutting plane algorithm solves the linear relaxation, finds cuts that violate the solution of this relaxation, resolves the linear relaxation and repeats this procedure until all integer variables have an integer solution. It was proved in [36, 63] that a pure integer programming problem can be solved by this procedure in a finite number of steps. However, this algorithm is not very useful in practice due to the tailing-off procedure [33], which results in a rapid decrease in the rate of progress towards the MIP solution as the solution to the linear relaxation approaches the MIP solution. However, this algorithm can be used to generate an improved formulation. Usually, a branch and bound algorithm can then solve this improved formulation faster than the original MIP. Hence, development of cutting planes is a topic that has been the subject of research attention for several decades [89].

Cutting planes can be classified into two types based on the class of problems for which they are generated [75, 89].

1. *Cuts for general MIP.* These cuts can be generated for any MIP, and do not utilize any underlying problem structure. They are usually generated for a relaxation of the MIP that has a simple structure. While the cuts may be very strong for the relaxation, their strength with respect to the MIP itself is quite hard to establish. Examples of these cuts are the cover inequalities and lifted cover inequalities [44], flow cover inequalities [102], and single-constraint lifting cuts [15, 16]. These cuts can be further classified as follows.
2. *Special structure cuts.* These cuts are generated by utilizing underlying problem structure, and can be very strong as they may be facets of the convex hull of feasible MIP solutions. However, they can be applied only to the class of problems for which they are generated. Some examples are cuts for the traveling salesman problem [69] and the set packing problem [100].

Cutting planes can also be classified in the following manner.

1. *Single-constraint cuts:* These cuts are obtained based on valid inequalities for relaxations of MIPs with only one constraint, or a linear combination of constraints. Some examples of these cuts are Gomory fractional cuts [62, 64], Gomory mixed integer cuts [63], disjunctive cuts [23], split cuts [42], lift-and-project cuts [24], n -step MIR cuts [78] and n -step mingling cuts [18].
2. *Multi-constraint cuts:* These cuts are obtained from MIP relaxations with multiple constraints. Several methods have been used to identify new multi-constraint cuts for MIPs. One of them is to study the facets of higher-dimensional infinite group polyhedra [3, 50, 51, 74] and use them to identify new cuts. The infinite

group problem [65, 66, 67, 68] is a relaxation of general MIP in an infinite-dimensional space. A class of functions referred to as extreme functions or facets for the infinite group problem [68] can be used to identify strong valid inequalities for general MIPs. Another approach is based on the notion of lattice-free intersection cuts [22]. Lattice-free intersection cuts are valid inequalities constructed for polyhedra after removing a lattice-free body (a polyhedron with no integer points in its interior). The split cuts [42] are intersection cuts obtained for one-row relaxations. Two-constraint intersection cuts have been studied from a theoretical perspective in [11, 31, 43], and from a computational perspective in [56]. Theoretical extensions on lattice-free intersection cuts have also been studied in [25, 37, 49, 59].

The valid inequalities generated for general MIPs in this dissertation are all multi-constraint cuts for general MIPs, and can also be customized to generate cuts for special structure problems.

II.1.2.3 Branch and Cut Algorithm

The branch and cut algorithm incorporates the main ideas of the branch and bound and the cutting plane algorithms into a single solution technique. This cutting plane algorithm generates cutting planes for the node problems in the branch and bound tree. Specifically, instead of directly solving the linear relaxation of an active node problem and branching, the branch and cut algorithm adds cutting planes and resolves the LP relaxation so as to develop a tighter approximation of the convex hull and accelerate progress towards a MIP solution.

Branch and cut does not have the disadvantages of branch and bound algorithms or cutting plane algorithms, as it uses cutting planes to generate tighter linear approxi-

mations of the node problems, and uses branching to create new nodes when tailing-off occurs due to the addition of cutting planes. Branch-and-cut was first introduced in [101]. Surveys on branch and cut algorithms are available in [55, 76, 92, 94]. Today, branch and cut is the most commonly used algorithm by commercial solvers to solve MIPs.

II.2 Mixed Integer Rounding

Mixed Integer Rounding (MIR) is a technique used to generate valid inequalities for general MIPs [90, 97, 117]. It was proved in [98] that MIR can generate all the facets of a general 0-1 MIP. MIR can also be used to obtain strong valid inequalities based on 1-row relaxations for general MIPs [90]. Dash and Günlük [46] proposed the 2-step MIR inequalities, which are generalization of the MIR Inequalities. The MIR inequalities and 2-step MIR inequalities are also facets for the infinite group problem [65, 66, 67, 68]. Other families related to MIR cuts for general MIPs are the split cuts [42] and disjunctive cuts [23]. Split cuts for a polyhedron $P = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p : Ax + Gy \geq b\}$ are obtained by considering the polyhedra $P^1 = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p : Ax + Gy \geq b, \pi x \geq \pi_0\}$ and $P^2 = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p : Ax + Gy \geq b, \pi x \leq \pi_0 - 1\}$ where (π, π_0) are integer valued. Disjunctive cuts [23] are obtained by considering subsets of a polyhedron, developing valid inequalities for these subsets and using them to generate new valid inequalities for the original polyhedron. It is shown in [98] that the MIR cuts, split cuts and disjunctive cuts are equivalent. We first present the MIR inequalities. Our presentation and notation closely follows [78].

The simplest form of the MIR inequality is defined for the set

$$Q^{1,1} = \{(y_1, v) \in \mathbb{Z} \times \mathbb{R}_+ : \alpha_1 y_1 + v \geq \beta\},$$

where $\alpha_1, \beta \in \mathbb{R}$, $\alpha_1 > 0$ and $\beta^{(1)} = \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor > 0$.

Theorem 11. [117] *The inequality*

$$\beta^{(1)}y_1 + v \geq \beta^{(1)} \left\lceil \frac{\beta}{\alpha_1} \right\rceil. \quad (2.1)$$

is valid for $Q^{1,1}$, and facet-defining for $\text{conv}(Q^{1,1})$.

The inequality (2.1) referred to as the 1-step MIR facet. MIR can also be used to generate strong valid inequalities for the general mixed integer knapsack set

$$Y_1 = \{(x_1, \dots, x_N, s) \in \mathbb{Z}_+^N \times \mathbb{R}_+ : \sum_{j=1}^N a_j x_j + s \geq b\}.$$

Let the set of indices $\{1, \dots, N\}$ be partitioned into two disjoint subsets J_0, J_1 . Given a parameter α_1 such that $b^{(1)} = b - \alpha_1 \lfloor b/\alpha_1 \rfloor > 0$, the defining inequality of Y_1 can be relaxed as follows:

$$\sum_{j \in J_0} \alpha_1 \left\lceil \frac{a_j}{\alpha_1} \right\rceil x_j + \sum_{j \in J_1} \left(\left\lfloor \frac{a_j}{\alpha_1} \right\rfloor + a_j^{(1)} \right) x_j + s \geq b. \quad (2.2)$$

To see that (2.2) is a relaxation, note that $\lceil a_j/\alpha_1 \rceil \geq \alpha_1$ and $a_j^{(1)} = a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor$. Hence, for $j \in J_0$, a_j is relaxed to $\alpha_1 \lceil a_j/\alpha_1 \rceil$ and for $j \in J_1$, a_j is replaced by $a_j^{(1)} + \lfloor a_j/\alpha_1 \rfloor$. This is possible as $x_j \geq 0$. Rearranging terms, (2.2) can be written as

$$\alpha_1 \left(\sum_{j \in J_0} \left\lceil \frac{a_j}{\alpha_1} \right\rceil x_j + \sum_{j \in J_1} \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor x_j \right) + \left(\sum_{j \in J_1} a_j^{(1)} x_j + s \right) \geq b. \quad (2.3)$$

Observe that the inequality (2.3) has a structure similar to the defining inequality of $Q^{1,1}$. In fact, the first sum with coefficient α_1 is an integer and the second group is non-negative. Hence, replacing y_1 and v by the corresponding sums from (2.3), we get the MIR inequality for the general mixed-integer knapsack set. To get the strongest inequality, the coefficients of the variables in the MIR inequality have to be minimized. It can be easily verified that this occurs when $J_0 = \{j : a_j^{(1)} \geq b^{(1)}\}$ and

$J_1 = \{j : a_j^{(1)} < b^{(1)}\}$. The MIR inequality can be written the following compact form.

Theorem 12. *The inequality*

$$\sum_{j=1}^N \mu_{(\alpha_1, b)}^1(a_j)x_j + s \geq \mu_{(\alpha_1, b)}^1(b) \quad (2.4)$$

is valid for Y_1 , where

$$\mu_{(\alpha_1, b)}^1 = b^{(1)} \lfloor t/\alpha_1 \rfloor + \min\{b^{(1)}, t^{(1)}\}.$$

Different variations of Theorem 12 can be found in [78, 90, 97, 117]. The inequality (2.4) is also facet-defining for $\text{conv}(K_{\geq})$ under certain additional conditions, and is referred to as the 1-step MIR inequality for Y_1 . When $\alpha_1 = 1$, the inequality (2.4) becomes the Gomory Mixed Integer Cut [63].

II.3 n -step MIR Inequalities

In this section, we briefly review the n -step MIR inequalities of Kianfar and Fathi [78]. We first describe some notation required to present these inequalities. For an $n \in \mathbb{N}$, let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_j > 0$, $j = 1, \dots, n$. For $\beta \in \mathbb{R}$ define the recursive remainders $\beta^{(j)} = \beta^{(j-1)} - \alpha_j \lfloor \beta^{(j-1)}/\alpha_j \rfloor$, where $\beta^{(0)} := \beta$. Note that $0 \leq \beta^{(j)} < \alpha_j$ for $j = 1, \dots, n$. We also assume that $\sum_a^b(\cdot) = 0$ and $\prod_a^b(\cdot) = 1$ whenever $a > b$.

The n -step MIR inequalities are generalization of the MIR inequalities, and are developed for general MIPs based on the facets of a certain $n + 1$ -dimensional set [78]. The simplest form of the n -step MIR inequality is a valid inequality for the set

$$Q^{1,n} = \{(y_1, \dots, y_n, v) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+ : \sum_{j=1}^n \alpha_j y_j + v \geq \beta\}.$$

The validity of the n -step MIR inequality for $Q^{1,n}$ requires the following conditions:

$$\alpha_j \lceil \beta^{(j-1)} / \alpha_j \rceil \leq \alpha_{j-1}, j = 2, \dots, n. \quad (2.5)$$

In order to get non-trivial inequalities, it is also assumed that $\beta^{(j-1)} / \alpha_j \notin \mathbb{Z}$, $j = 1, \dots, n$.

Theorem 13. [78] *If conditions (2.5) hold, the inequality*

$$\beta^{(n)} \sum_{j=1}^n \prod_{l=j+1}^n \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil y_j + v \geq \beta^{(n)} \prod_{l=1}^n \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil. \quad (2.6)$$

is facet-defining for $\text{conv}(Q^{1,n})$.

The inequality (2.6) is referred to as the n -step MIR facet for $Q^{1,n}$. An intermediate result from [78], which will be useful for our results, is that the inequalities

$$\alpha_j \left(\sum_{i=1}^j \prod_{l=i+1}^j \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil y_i - \prod_{l=1}^j \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil + \left\lceil \frac{\beta^{(j-1)}}{\alpha_j} \right\rceil \right) + \sum_{i=j+1}^n \alpha_i y_i + v \geq \beta^{(j-1)}; \quad j = 1, \dots, n \quad (2.7)$$

are also valid for $Q^{1,n}$ if conditions (2.5) are satisfied.

The n -step MIR facet can be used to generate strong valid inequalities for the general mixed integer knapsack set Y_1 . This requires n parameters $(\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha_j > 0$ for $j = 1, \dots, n$, the conditions

$$\alpha_j \lceil b^{(j-1)} / \alpha_j \rceil \leq \alpha_{j-1}, j = 2, \dots, n \quad (2.8)$$

are satisfied and $b^{(n)} > 0$.

Let the set of indices of integer variables in Y_1 , $\{1, \dots, N\}$ be partitioned into $n + 1$ disjoint subsets J_0, \dots, J_n . Based on the n parameters $\alpha_1, \dots, \alpha_n$, the defining inequal-

ity of Y_1 can be relaxed as follows:

$$\sum_{m=0}^{n-1} \sum_{j \in J_m} \left(\sum_{i=1}^m \alpha_i \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor + \alpha_{m+1} \left\lfloor \frac{a_j^{(m)}}{m+1} \right\rfloor \right) x_j + \sum_{j \in J_n} \left(\sum_{i=1}^n \alpha_i \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor + a_j^{(n)} \right) x_j \geq b. \quad (2.9)$$

To see that (2.9) is a relaxation, observe that for $1 \leq m \leq n$,

$$a_j = \sum_{i=1}^m \alpha_i \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor + a_j^{(m)}.$$

This identity has been used to replace a_j for $j \in J_n$, and to relax a_j for $j \in J_m$, $m = 0, \dots, n-1$ as $\alpha_{m+1} \left\lfloor a_j^{(m)} / \alpha_m \right\rfloor \geq a_j^{(m)}$ and $x_j \geq 0$. Rearranging the terms of (2.9), we get

$$\sum_{i=1}^n \alpha_i \left(\sum_{j \in J_{i-1}} \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor x_j + \sum_{m=i}^n \sum_{j \in J_m} \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor x_j \right) + \left(\sum_{j \in J_n} a_j^{(n)} x_j + s \right) \geq b. \quad (2.10)$$

It can be easily verified that in the inequality (2.10), the expression with coefficient α_1 is an integer, the expressions with coefficients α_j , $j = 2, \dots, n$ are non-negative integers and the sum $\sum_{j \in J_n} a_j^{(n)} x_j + s$ is non-negative. These expressions match the non-negativity and integrality conditions of the variables y_1, \dots, y_n, v in $Q^{1,n}$. Therefore, the variables y_1, \dots, y_n, v in the n -step MIR facet (2.6) can be replaced with the expressions in (2.10) to get a valid inequality for Y_1 . The inequality obtained using this procedure is the n -step MIR inequality for Y_1 . The n -step MIR inequality is strongest when the coefficients of the integer variables are minimized. It can be easily verified that this occurs when $J_m = \{j : a_j^{(k)} < b^{(k)}, k = 1, \dots, m, a_j^{(m+1)} \geq b^{(m+1)}\}$, $m = 0, \dots, n-1$, $J_n = \{j : a_j^{(k)} < b^{(k)}, k = 1, \dots, n\}$ [78]. Based on this, the n -step MIR inequality can be written using a function of the coefficients a_j that is defined by the parameters $(\alpha_1, \dots, \alpha_n)$ and the right-hand side b . This function is referred to as the n -step MIR function, and is defined as follows:

Definition 14. [18, 78] For $n \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$ such that $\alpha_j > 0$, $j = 1, \dots, n$ and right-hand side $b \in \mathbb{R}$, the n -step MIR function is

$$\mu_{\alpha,b}^n(t) = \begin{cases} \sum_{k=l}^m \prod_{l=k+1}^n \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{t^{(k-1)}}{\alpha_k} \right\rfloor b^{(n)} + \prod_{l=m+2}^n \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{t^{(m)}}{\alpha_{m+1}} \right\rfloor b^{(n)}, \\ \quad t \in I_m^n; m = 0, \dots, n-1 \\ \sum_{k=l}^n \prod_{l=k+1}^n \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{t^{(k-1)}}{\alpha_k} \right\rfloor b^{(n)} + t^{(n)}, \quad t \in I_n^n, \end{cases}$$

where $t^{(j)} = t^{(j-1)} - \alpha_j \lfloor t^{(j-1)} / \alpha_j \rfloor$, $j = 1, \dots, n$, $t^{(0)} = t$ and the sets I_0^n, \dots, I_n^n are defined as

$$I_m^n = \{t \in \mathbb{R} : t^{(k)} < b^{(k)}, k = 1, \dots, m, t^{(m+1)} \geq b^{(m+1)}\};$$

$$I_n^n = \{t \in \mathbb{R} : t^{(k)} < b^{(k)}, k = 1, \dots, n\}.$$

We present the n -step MIR inequality for Y_1 in a compact form using the n -step MIR function in the following Theorem.

Theorem 15. [78] If conditions (2.8) hold, the inequality

$$\sum_{j=1}^N \mu_{\alpha,b}^n(a_j) x_j + s \geq \mu_{\alpha,b}^n(b) \quad (2.11)$$

is valid for Y_1 .

In other words, the n -step MIR inequality is obtained by applying the n -step MIR function on a_j 's and b . A variant of the n -step MIR inequalities are the n -step mingling inequalities of Atamtürk and Kianfar [18] that are obtained by incorporating the upper bounds of integer variables into the n -step MIR inequalities.

II.4 Mixing Inequalities

We now review the mixing inequalities [70]. Günlük and Pochet studied the mixing set

$$Q^{m,1} = \{(y^1, \dots, y^m, v) \in \mathbb{Z}^m \times \mathbb{R}_+ : \alpha_1 y^i + v \geq \beta_i, i = 1, \dots, m\}.$$

(where $\alpha_1 > 0$, and the superscripts of Q denote the number of rows and integer variables in each row, respectively) [70]. The 1-step MIR inequality [70, 97, 117] for the inequality i in $Q^{m,1}$ can be written as

$$v \geq \beta_i^{(1)} (\lceil \beta_i / \alpha_1 \rceil - y^i). \quad (2.12)$$

Consider a non-empty $K \subseteq M$. To simplify the notation and without loss of generality we assume $K = \{1, \dots, k\}$ and $\beta_{i-1}^{(1)} \leq \beta_i^{(1)}, i = 2, \dots, k$. By *mixing* the 1-step MIR inequalities (2.12) for $i \in K$, Günlük and Pochet [70] presented the *mixed MIR inequalities* for $Q^{m,1}$.

Theorem 16. [70] *The inequalities*

$$v \geq \sum_{i=1}^k (\beta_i^{(1)} - \beta_{i-1}^{(1)}) \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y^i \right) \quad (2.13)$$

$$v \geq \sum_{i=1}^k (\beta_i^{(1)} - \beta_{i-1}^{(1)}) \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y^i \right) + (\alpha_1 - \beta_k^{(1)}) \left(\left\lceil \frac{\beta_1}{\alpha_1} \right\rceil - y^1 - 1 \right). \quad (2.14)$$

are valid for $Q^{m,1}$.

where $\beta^{(1)} = 0$ by definition. We refer to (2.13) and (2.14) as the *type I* and *type II* mixed MIR inequalities generated by K , respectively. It is shown in [70] that the convex hull of $Q^{m,1}$ is completely described by inequalities of the form (2.13) and (2.14) generated by all possible subsets K of M .

Variations of the mixing set have also been studied: The mixing set with divisible capacities, i.e. $\{(y^1, \dots, y^m, v) \in \mathbb{Z}^m \times \mathbb{R}_+ : \alpha_1^i y^i + v \geq \beta_i, i = 1, \dots, m\}$ where $\alpha_1^m | \alpha_1^{m-1} | \dots | \alpha_1^1$, was studied in [41] for $m = 2$ and in [39, 119] for general m . A simple algorithm for linear optimization over this set along with a compact extended formulation for it are devised in [40]. The case where the capacities are not divisible was studied in [5] for $m = 2$. Other variants of the mixing set include the continuous mixing

set [113, 120], the mixing set with flows [38] and the mixing set linked by bidirected paths [53]. The mixing inequalities were studied from a group-theoretic perspective in [52]. Bounds on the MIR rank of the mixing inequalities have been proposed in [47, 48].

The mixing inequalities can also be used to generate valid inequalities for special structure MIPs, as shown in [70]. We briefly review the valid inequalities generated for lot sizing and facility location problems presented in [70].

II.4.1 Valid Inequalities for Production Planning Problems

Let T be the set of time periods with $|T| = m$ and C be the production capacity. In the constant capacity single item lot-sizing problem (LCC), the goal is to find a production plan that minimizes the sum of production, inventory and setup costs over all periods while meeting demand (without backlogging) and satisfying capacity constraints. Let x_t be the production, and s_t be the inventory at the end of period t . Let y_t be a binary variable that takes a value of 1 if production occurs in period t and 0 otherwise. The feasible region of LCC, denoted by X^{CCL} is defined as

$$X^{CCL} = \left\{ (x, s, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{B}^m : \right. \\ \left. \begin{aligned} s_{t-1} + x_t &= d_t + s_t, & t \in T \\ x_t &\leq Cy_t & t \in T \end{aligned} \right\}, \quad (2.15)$$

$$\left. \begin{aligned} s_{t-1} + x_t &= d_t + s_t, & t \in T \\ x_t &\leq Cy_t & t \in T \end{aligned} \right\}, \quad (2.16)$$

where $s_0 = s_n = 0$. Günlük and Pochet [70] showed that by aggregation of the flow balance constraints (2.15) and relaxing x_t variables to their upper bounds Cy_t , inequalities with a structure similar to the defining inequalities of $Q^{m,1}$ can be constructed. The mixing procedure can then be applied on these inequalities to get mixing-type inequalities for X^{CCL} . We present the construction of these *base inequalities* for which the

mixing inequalities will be written. We follow the notation of [104] as much as possible. For any $k, l \in T$, where $k < l$, let $S \subseteq \{k, \dots, l\}$. For $i \in S$, let $S_i = S \cap \{k, \dots, i\}$ and $b_i = \sum_{t=k}^{n_i-1} d_t$, where

$$n_i = \begin{cases} \min\{t : t \in S \setminus S_i\}, & \text{if } S \setminus S_i \neq \emptyset \\ l + 1, & \text{if } S \setminus S_i = \emptyset. \end{cases} \quad (2.17)$$

Adding equalities (2.15) from periods k to $n_i - 1$, we get

$$s_{k-1} + \sum_{t=k}^{n_i-1} x_t = b_i + s_{n_i-1}. \quad (2.18)$$

Note that $S_i \subseteq \{k, \dots, n_i - 1\}$ by definition. If we relax x_t , $t \in S_i$, in (2.18) to its upper bound based on (2.16) and drop $s_{n_i-1} (\geq 0)$, we get the following valid inequality:

$$s_{k-1} + \sum_{t \in \{k, \dots, n_i-1\} \setminus S_i} x_t + C \sum_{t \in S_i} y_t \geq b_i. \quad (2.19)$$

Setting $v_i := s_{k-1} + \sum_{t \in \{k, \dots, n_i-1\} \setminus S_i} x_t$, inequality (2.19) becomes

$$C \sum_{t \in S_i} y_t + v_i \geq b_i, \quad (2.20)$$

which is of the same form as the defining inequalities of $Q^{m,1}$ (notice that $v_i \in \mathbb{R}_+$, $z_i \in \mathbb{Z}_+$). Considering $I \subseteq S$, we get an inequality like (2.20) for each $i \in I$. The mixing procedure can be applied on these base inequalities to get new valid inequalities for X^{CCL} . In fact, it is observed in [70] that these inequalities are precisely the (k, l, S, I) inequalities of Pochet and Wolsey [104].

II.4.2 Valid Inequalities for Capacitated Facility Location (CFL) Problems

Let $P := \{1, \dots, n_P\}$ be a set of potential facilities with capacity C , $Q := \{1, \dots, n_Q\}$ be a set of clients with demands d_q , $q \in Q$. The single capacity facility location problem

aims to minimize the setup and distribution costs for facilities while satisfying customer demand. The feasible region of CFL, denoted by X^{CCF} is defined as

$$X^{CCF} = \left\{ (x, u) \in \mathbb{R}_+^{n_P n_Q} \times \mathbb{B}_P^n : \right. \\ \left. \sum_{p \in P} x_{pq} = d_q, \quad q \in Q \right. \quad (2.21)$$

$$\left. \sum_{q \in Q} v_{jk} \leq C y_p, \quad p \in P \right\}. \quad (2.22)$$

Let $I = \{1, 2, \dots, n_I\}$ and for $i \in I$ choose $S_i \subseteq P$ and $K_i \subseteq Q$. Let $b_i := \sum_{q \in K_i} d_q$ be the total demand of clients in K_i . Adding the demand constraints (2.21) for $q \in K_i$, we get

$$\sum_{p \in P} w_p^i = b_i \quad (2.23)$$

where $w_p^i = \sum_{q \in K_i} x_{pq}$ is the total demand of clients in K_i satisfied by facility p . Now, by (2.22), we have $w_p^i \leq C y_p$. Therefore, for $p \in S_i$, the variables w_p^i in (2.23) can be relaxed to its upper bound to get

$$\sum_{p \in P \setminus S_i} w_p^i + C \sum_{p \in S_i} y_p \geq b_i, \quad i \in I. \quad (2.24)$$

As the inequalities (2.24) have a structure similar to the base inequalities of $Q^{m,1}$, the mixing procedure can be applied on these inequalities to obtain new valid inequalities for X^{CFL} [70]. As observed in [70], in the special case that the sets $\{S^i, K^i\}$ form a nested family, i.e. $K^i \subset K^{i+1}$ and $S^i \subset S^{i+1}$, the mixing-type inequalities obtained for X^{CCF} are simply the valid inequalities obtained in [1] for the capacitated facility location problem.

II.5 Conic MIR Inequalities

A second-order conic mixed-integer program is formulated as

$$\begin{aligned}
 & \min \quad cx + ry \\
 & \text{s.t.} \quad \|A_i x + G_i z - b_i\| \leq d_i x + e_i z - h_i, \quad i = 1, \dots, k, \\
 & \quad x \in \mathbb{Z}^n, z \in \mathbb{R}^p,
 \end{aligned} \tag{SOCMIP}$$

where A_i , G_i , and b_i have m_i rows, d_i , e_i , c and r are vectors of appropriate dimensions and h_i is a scalar. It is assumed that all the data is rational. We refer the reader to [6, 26, 32, 84, 99] for detailed coverage of conic optimization and SOCP.

Solution methods for nonlinear integer programming can be used to solve SOCMIP. One group of these methods use the SOCP relaxation of the problem in a branch-and-bound procedure [29, 30, 71, 83, 109]. Other methods use the polyhedral relaxation of nonlinear constraints of the SOCMIP. This relaxation is constantly updated within the course of solving a master problem or inside a branch-and-bound framework. Outer approximation [54, 57], generalized Benders' decomposition [61], LP/NLP-based branch-and-bound [105], the extended cutting plane method [116], as well as methods used in [4, 29, 111, 112, 114] are examples of such methods. Cuts that result in stronger linear or conic relaxations of the feasible region are of interest in all these methods. Generalization of lift-and-project and reformulation-linearization technique (RLT) to non-convex optimization [107, 109, 110], and hierarchies of semidefinite relaxations proposed for non-convex sets defined by quadratic functions [79] and nonlinear 0-1 programs [82], are examples of stronger relaxations. Çezik and Iyengar [35] developed valid inequalities for conic mixed integer sets in a procedure that uses Chvátal-Gomory or mixed integer rounding (MIR) cuts [90, 97, 117]. Atamtürk and Narayanan [20] presented lifting of conic inequalities for conic mixed integer programs.

They used a polyhedral reformulation of a conic constraint of SOCMIP in a higher dimensional space to develop conic MIR inequalities. More specifically, they reformulated the mixed integer second-order conic set

$$X = \{(x, z) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : \|Ax + Gz - b\| \leq dx + ez - h\}$$

as

$$t_0 \leq dx + ez - h \tag{2.25}$$

$$t_i \geq |a_i x + g_i z - b|, \quad i = 1, \dots, m \tag{2.26}$$

$$t_0 \geq \|t\| \tag{2.27}$$

$$(x, z, t, t_0) \in \mathbb{Z}_+^n \times \mathbb{R}_+^{p+m+1}, \tag{2.28}$$

where a_i and g_i , $i = 1, \dots, m$, are the m rows of A and G , respectively. They referred to constraints of the form (2.26) as second-order polyhedral conic constraints. They first studied a simple set defined by a constraint of the form (2.26), i.e.

$$\overline{Q} := \{(x, w^+, w^-, t) \in \mathbb{Z} \times \mathbb{R}_+^3 : |x + w^+ - w^- - \beta| \leq t\},$$

and developed a linear inequality, referred to as the *simple conic MIR inequality* for this set, and proved that along with the defining inequality, it describes the convex hull of \overline{Q} .

Lemma 17. [19] *The simple conic MIR inequality*

$$(1 - 2f)(x - \lfloor b \rfloor) + f \leq t + w^+ + w^- \tag{2.29}$$

is valid for \overline{Q} and cuts off all points in $\text{relax}(\overline{Q}) \setminus \text{conv}(\overline{Q})$.

In the above lemma, $f = b - \lfloor b \rfloor$, $\text{relax}(\overline{Q})$ is the continuous relaxation of \overline{Q} obtained by dropping the integrality condition on x and $\text{conv}(\overline{Q})$ is the convex hull of \overline{Q} .

They also developed a nonlinear valid inequality for \overline{Q} based on the simple conic MIR inequality. Specifically, they observed that the inequality

$$|(1 - 2f)(x - \lfloor b \rfloor) + f| \leq t + w^+ + w^- \quad (2.30)$$

is also valid for \overline{Q} . PSOC inequalities such as (2.30) are of interest because they can be used to define nonlinear inequalities as explained in [19]: Similar to the reformulation of X , a second-order conic set like $X_I = \{(x, t) \in \mathbb{Z}^N \times \mathbb{R} : \|Ax - b\| \leq t_0\}$ can be reformulated as

$$t_i \geq |a^i x - b_i|, \quad i = 1, \dots, m \quad (2.31)$$

$$t_0 \geq \sqrt{t_1^2 + \dots + t_m^2} \quad (2.32)$$

$$(x, t, t_0) \in \mathbb{Z}^N \times \mathbb{R}^{m+1}. \quad (2.33)$$

Now if a PSOC inequality like

$$|\pi^i x - \pi_0^i| \leq t_i \quad (2.34)$$

is generated for each base inequality i in (2.31), based on (2.32), we can write

$$t_0 \geq \sqrt{\sum_{i=1}^m (\pi^i x - \pi_0^i)^2}, \quad (2.35)$$

which is a nonlinear conic inequality.

They then used the simple conic MIR inequality to develop the *conic MIR function*, which can be used to generate *conic MIR inequalities* for the set

$$S := \left\{ (x, z^+, z^-, t) \in \mathbb{Z}_+^N \times \mathbb{R}_+^3 : \left| \sum_{j=1}^N a_j x_j + z^+ - z^- - b \right| \leq t \right\}$$

(the defining constraint of S can be obtained from an inequality of the form (2.26) after aggregating its positive and negative continuous parts into z^+ and z^- , respectively).

Definition 18. [19] For $0 \leq f < 1$ let the conic MIR function $\phi_f : \mathbb{R} \rightarrow \mathbb{R}$ be

$$\phi_f(a) = \begin{cases} (1 - 2f) \lfloor a \rfloor - f_a, & f_a < f, \\ (1 - 2f) \lfloor a \rfloor + f_a - 2f, & f \leq f_a \end{cases}$$

where $f_a = a - \lfloor a \rfloor$.

Theorem 19. [19] For any $\alpha \neq 0$ the conic MIR inequality

$$\sum_{j=1}^n \phi_{f_\alpha}(a_j/\alpha)x_j - \phi_{f_\alpha}(b/\alpha) \leq (t + z^+ + z^-)/|\alpha| \quad (2.36)$$

where $f_\alpha = b/\alpha - \lfloor b/\alpha \rfloor$ is valid for S . Moreover, if α is chosen such that $\alpha = a_j$ and $b/a_j > 0$ for some $j \in \{1, \dots, n\}$ and $a_j \leq b$ for all $i \in \{1, \dots, n\} \setminus \{j\}$, then (2.36) is facet-defining for $\text{conv}(S)$.

They showed that the conic MIR inequality for S can also be used to derive a nonlinear conic MIR inequality for the set X based on the reformulation (2.25)-(2.28). Moreover, they also developed cuts for linear MIPs using the conic MIR inequality. As observed in [19], any two linear constraints $c_1x \leq b_1$ and $c_2x \leq b_2$ can be equivalently written as the following PSOC constraint:

$$\left| \frac{c_1 - c_2}{2}x - \frac{b_1 - b_2}{2} \right| \leq \frac{b_1 + b_2}{2} - \frac{c_1 + c_2}{2}x. \quad (2.37)$$

As a result, the conic MIR function can be used to generate a valid inequality for the feasible set of this pair. In particular, using this technique, Atamtürk and Narayanan [19] showed that the well-known MIR inequality [90] is a conic MIR inequality.

CHAPTER III

MIXED n -STEP MIR INEQUALITIES

In this chapter, we introduce new classes of multi-row valid inequalities for general and special structure linear MIPs, establish several theoretical properties and the computational effectiveness of these valid inequalities. More specifically, we study the following generalized mixing set

$$Q^{m,n} = \{(y^1, \dots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+ : \sum_{j=1}^n \alpha_j y_j^i + v \geq \beta_i, i = 1, \dots, m\}.$$

We refer to $Q^{m,n}$ as the n -mixing set. Note that the superscript m denotes the number of constraints, and n the number of integer variables in each constraint. We show that the idea of mixing can be generalized to n -step MIR inequalities. We develop the *type I and type II mixed n -step MIR inequalities* for the n -mixing set $Q^{m,n}$ under the condition that for each constraint i of $Q^{m,n}$ used in the mixing, α_j 's and β_i satisfy the same conditions required for validity of the n -step MIR inequality, i.e. $\alpha_j \left\lceil \beta_i^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}, j = 2, \dots, n$ (Section III.1). The mixed MIR inequalities of [70] simply correspond to the special case of $n = 1$. We then demonstrate the strength of the mixed n -step MIR inequalities by showing that the type I mixed n -step MIR inequalities define facets for the convex hull of $Q^{m,n}$, denoted by $\text{conv}(Q^{m,n})$, and type II mixed n -step MIR inequalities define faces of dimension at least $n(m-1)$ for $\text{conv}(Q^{m,n})$ and are facet-defining for this set if some additional conditions are satisfied (Section III.2).

We then show how the mixed n -step MIR inequalities for $Q^{m,n}$ can be used to generate mixed n -step MIR inequalities for the general multi-constraint mixed integer

set

$$Y_m = \{(x_1, \dots, x_N, s) \in \mathbb{Z}_+^N \times \mathbb{R}_+^m : \sum_{t \in T} a_{it}x_t + s_i \geq b_i, i = 1, \dots, m\},$$

where $T = \{1, \dots, N\}$ and $a_{it}, b_i \in \mathbb{R}$ for all i and t (Section III.3). Note that any set defined by m mixed integer constraints can be relaxed to a set of the form Y_m (see Section III.3). As a result, for a general MIP, the mixed n -step MIR generates valid inequalities that are based on multiple constraints. A mixed n -step MIR inequality for Y_m has n positive parameters, namely $\alpha_1, \dots, \alpha_n$, which must satisfy the n -step MIR conditions, i.e. $\alpha_j \left\lceil b_i^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}, j = 2, \dots, n$, for any constraint i of Y_m that is used in generating the inequality. Any set of values for the parameters $\alpha_1, \dots, \alpha_n$ that satisfy these conditions give a corresponding mixed n -step MIR inequality for Y_m . Notice that for validity of the mixed n -step MIR inequality for Y_m , no conditions on the coefficients a_{it} in Y_m is required. In other words, the restriction of n -step MIR conditions is only on the parameters of the cut, i.e. $\alpha_1, \dots, \alpha_n$, and as we will see in Section III.3, there are always infinitely many choices for these parameters that satisfy the n -step MIR conditions.

Next, we introduce a generalization of the constant capacity lot-sizing problem discussed in Section III.4.1, which we refer to as the multi-module lot-sizing problem. We show that the mixed n -step MIR inequalities can be used to generate valid inequalities for this problem. In MML, the total capacity in each period is the summation of integer multiples of several modules of different capacities. The mixed n -step MIR inequalities for MML generalize the valid inequalities discussed in Section III.4.1 and the (k, l, S, I) inequalities for the constant-capacity lot-sizing problem (CCL) [70, 104]. Similarly, we also introduce a generalization of the capacitated facility location problem discussed in Section III.4.1, which we refer to as the multi-module facility location problem (MMF), and show that the mixed n -step MIR inequalities can be used to

generate valid inequalities for this problem. The mixed n -step MIR inequalities for MMF generalize the mixed MIR inequalities for the constant-capacity facility location problem (CCF) [1, 2, 70] (Section III.4).

Finally, we provide our preliminary computational results on using the mixed n -step MIR inequalities in solving small MIPLIB instances as well as a set of MML instances (Section III.5). These results are quite promising in light of the fact that MIPLIB instances are notorious with respect to gap improvement beyond what is achieved by 1-step MIR [58]. Our results for MML instances show that mixed n -step MIR cuts are very efficient cutting planes for MML problems. The addition of mixed n -step MIR cuts results in a considerable reduction in integrality gap, and a decrease of several orders of magnitude in both solution time and number of nodes.

We also note that in the special case where the parameters α_j , $j = 1, \dots, n$, in $Q^{m,n}$ are *divisible*, i.e. $\alpha_n | \alpha_{n-1} | \dots | \alpha_1$, the validity conditions of the n -step MIR are always satisfied. Consequently, all results in this chapter are always true for the special case of divisible parameters (as we will see in Section III.4, in the case of MML and MMF, the parameters α_j , $j = 1, \dots, n$, are the capacities of modules).

III.1 Mixed n -step MIR Inequalities for the n -mixing Set

In this section, we show that mixing can be generalized to the n -step MIR inequalities. In other words, one can mix the n -step MIR inequalities written for the individual constraints of the n -mixing set $Q^{m,n}$ and get a valid inequality based on multiple constraints (called the mixed n -step MIR inequality) for this set. Any subset of constraints of $Q^{m,n}$ can be chosen to be mixed. Let $K \subseteq M$ denote the index set of the chosen constraints. To simplify the notation and without loss of generality throughout the chapter we assume $K = \{1, \dots, k\}$ and $\beta_{i-1}^{(n)} \leq \beta_i^{(n)}$, $i = 2, \dots, k$. Also note that

according to (2.8), for the n -step MIR inequality to be valid for each *base* constraint i , $i \in K$, the conditions

$$\alpha_j \left\lceil \beta_i^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}, \quad j = 2, \dots, n, i \in K \quad (3.1)$$

must be satisfied (as mentioned, the assumptions $\beta_i^{(j-1)} / \alpha_j \notin \mathbb{Z}$, $j = 1, \dots, n, i \in K$ are also required to avoid trivial inequalities). Now assuming (3.1) holds, the n -step MIR inequality (2.6) written for constraint i of $Q^{m,n}$, $i \in K$, is valid for $Q^{m,n}$ and can be written as

$$v \geq \beta_i^{(n)} \left(\prod_{l=1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^n \prod_{l=j+1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_j^i \right). \quad (3.2)$$

To simplify notation in the rest of the chapter, we define the function $\phi^i : \mathbb{Z}^n \rightarrow \mathbb{Z}$ to denote the integer-valued expression inside the parentheses in (3.2) and refer to it as the n -mixing function, i.e.

$$\phi^i(y^i) := \prod_{l=1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^n \prod_{l=j+1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_j^i \quad \text{for } i \in K. \quad (3.3)$$

Note that ϕ^i is a function of variables $y^i = (y_1^i, \dots, y_n^i)$ which depends on parameters α and β_i . Now the n -step MIR inequality (3.2) can be written as

$$v \geq \beta_i^{(n)} \phi^i(y^i). \quad (3.4)$$

We show that inequalities (3.4), $i \in K$, can be mixed to obtain the following valid inequalities for $Q^{m,n}$:

$$v \geq \sum_{i=1}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i), \quad (3.5)$$

$$v \geq \sum_{i=1}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) + \left(\alpha_n - \beta_k^{(n)} \right) (\phi^1(y^1) - 1), \quad (3.6)$$

where $\beta_0^{(n)} = 0$ by definition. We refer to (3.5) and (3.6) as the *type I and type II mixed n -step MIR inequalities*, respectively. The validity of (3.5) and (3.6) can be proved using an argument similar to the one used in [70] for validity of (2.13) and (2.14) but requires an additional lemma:

Lemma 20. *For $i \in K$, the inequality*

$$v \geq \beta_i^{(n)} + \alpha_n (\phi^i(y^i) - 1) \quad (3.7)$$

is valid for $Q^{m,n}$.

Proof. For $i \in K$, since (3.1) holds, inequality (2.7) written for the constraint i of $Q^{m,n}$ and $j = n$, i.e.

$$\alpha_n \left(\sum_{i=1}^n \prod_{l=i+1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_i - \prod_{l=1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil + \left\lceil \frac{\beta_i^{(n-1)}}{\alpha_n} \right\rceil \right) + v \geq \beta_i^{(n-1)} \quad (3.8)$$

is valid for $Q^{m,n}$. By subtracting $\alpha_n \left\lfloor \beta_i^{(n-1)} / \alpha_n \right\rfloor$ from both sides and re-arranging the terms we get (3.7). \square

Theorem 21. *If conditions (3.1) hold, the type I and type II mixed n -step MIR inequalities (3.5) and (3.6) are valid for $Q^{m,n}$.*

Proof. To prove the validity of (3.5), consider a fixed point $(\hat{y}^1, \dots, \hat{y}^m, \hat{v}) \in Q^{m,n}$. Define $\lambda := \max_{i \in K} \phi^i(\hat{y}^i)$ and $p := \max\{i \in K : \phi^i(\hat{y}^i) = \lambda\}$. If $\lambda \leq 0$, then it is trivial that (3.5) is satisfied because $\hat{v} \geq 0$, and by the assumed ordering of indices in K , $\beta_i^{(n)} - \beta_{i-1}^{(n)} \geq 0$, $i \in K$. If $\lambda \geq 1$, then since $\phi^i(\hat{y}^i)$ is an integer, we can write

$$\begin{aligned} \sum_{i=1}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(\hat{y}^i) &\leq \sum_{i=1}^p (\beta_i^{(n)} - \beta_{i-1}^{(n)}) \lambda + \sum_{i=p+1}^k (\beta_i^{(n)} - \beta_{i-1}^{(n)}) (\lambda - 1) \\ &= \beta_p^{(n)} (\lambda) + \left(\beta_k^{(n)} - \beta_p^{(n)} \right) (\lambda - 1) \\ &= \beta_p^{(n)} + \beta_k^{(n)} (\lambda - 1) \end{aligned}$$

$$\begin{aligned}
&\leq \beta_p^{(n)} + \alpha_n(\lambda - 1) \\
&= \beta_p^{(n)} + \alpha_n(\phi^p(\hat{y}^p) - 1) \leq \hat{v}
\end{aligned}$$

The last inequality follows from Lemma 20. This proves the validity of (3.5). The validity of (3.6) can be proved very similarly. \square

Note that for $n = 1$ this proof reduces to the proof of validity of the mixed 1-step MIR inequalities in [70], where Lemma 20 was not required because for $n = 1$ inequality (3.7) simply reduces to the base inequality $\alpha_1 y_1^i + v \geq \beta_i$.

Consider the following generalization of $Q^{m,n}$ which has different continuous variables in each row:

$$\hat{Q}^{m,n} = \{(y^1, \dots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^m : \sum_{j=1}^n \alpha_j y_j^i + v_i \geq \beta_i, i = 1, \dots, m\}.$$

Let the variable $\bar{v} \in \mathbb{R}_+$ be such that $\bar{v} \geq v_i$ for all $i \in K$. Then as a direct result of Theorem 21, we have the following:

Corollary 22. *If conditions (3.1) hold, the mixed n -step MIR inequalities*

$$\bar{v} \geq \sum_{i=1}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) \quad (3.9)$$

$$\bar{v} \geq \sum_{i=1}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) + \left(\alpha_n - \beta_k^{(n)} \right) (\phi^1(y^1) - 1) \quad (3.10)$$

are valid for $\hat{Q}^{m,n}$. \square

Remark 1. (Divisible coefficients) An interesting special case of the n -mixing set $Q^{m,n}$ is when the coefficients are divisible, i.e. $\alpha_j | \alpha_{j-1}$, $j = 2, \dots, n$. Note that in this case for any $i \in K$ and $j \in \{2, \dots, n\}$, by definition of $\beta_i^{(j-1)}$, we have $\alpha_{j-1}/\alpha_j \geq \beta_i^{(j-1)}/\alpha_j$, which implies $\alpha_{j-1}/\alpha_j \geq \left\lceil \beta_i^{(j-1)}/\alpha_j \right\rceil$ because α_{j-1}/α_j is an integer. That means in this case conditions (3.1) are automatically satisfied. Consequently, all results

in this chapter are always true for the case where the elements of the parameter vector α are divisible, i.e. $\alpha_j | \alpha_{j-1}$, $j = 2, \dots, n$. \square

III.2 Facets Defined by Mixed n -step MIR Inequalities

In this section, we prove that the type I mixed n -step MIR inequalities define facets for $\text{conv}(Q^{m,n})$. We also show that the type II inequalities define faces of dimension at least $n(m-1)$ for $\text{conv}(Q^{m,n})$ and define facets for this set if some extra conditions on parameters are satisfied. These results demonstrate the strength and importance of these inequalities. Note that $\text{conv}(Q^{m,n})$ is non-empty and full-dimensional (is of dimension $mn+1$). That is because a point $P = (y^1, \dots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}^+$ with sufficiently large coordinates is feasible to $Q^{m,n}$ (since $\alpha_j > 0, j \in J$) and $P + e \in Q^{m,n}$ for all unit vectors $e \in \mathbb{R}^{mn+1}$. To prove the facet-defining property of the type I mixed n -step MIR inequality, we need to define some points and prove some properties for them first.

Definition 23. For $i \in M, t = 1, \dots, n$, define the points $p^{i,t} = (p_1^{i,t}, \dots, p_n^{i,t}) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$ such that

$$p_j^{i,t} = \begin{cases} \lfloor \beta_i^{(j-1)} / \alpha_j \rfloor & \text{for } j = 1, \dots, t-1 \\ \lceil \beta_i^{(j-1)} / \alpha_j \rceil & \text{for } j = t \\ 0 & \text{for } j = t+1, \dots, n, \end{cases}$$

and for $i \in K, t = 1, \dots, n$, define the points $q^{i,t} = (q_1^{i,t}, \dots, q_n^{i,t}) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$ such that

$$q_j^{i,t} = \begin{cases} \lfloor \beta_i^{(j-1)} / \alpha_j \rfloor & \text{for } j = 1, \dots, t \\ 0 & \text{for } j = t+1, \dots, n. \end{cases}$$

Lemma 24. The point $P = (\hat{y}^1, \dots, \hat{y}^m, \hat{v}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$ satisfies constraint i

of $Q^{m,n}$ if any of the following is true:

- (a). $i \in M$ and $\hat{y}^i = p^{i,t}$ for some $t \in \{1, \dots, n\}$,
- (b). $i \in K$ and $\hat{y}^i = q^{i,t}$ for some $t \in \{1, \dots, n\}$ and $\hat{v} \geq \beta_i^{(t)}$.

Proof. If (a) is true, by substituting the point P in constraint i of $Q^{m,n}$, we get $\sum_{j=1}^{t-1} \alpha_j \left\lfloor \beta_i^{(j-1)} / \alpha_j \right\rfloor + \alpha_t \left\lfloor \beta_i^{(t-1)} / \alpha_t \right\rfloor + \hat{v} \geq \beta_i$, or $\alpha_t \left\lfloor \beta_i^{(t-1)} / \alpha_t \right\rfloor + \hat{v} \geq \beta_i^{(t-1)}$, which is trivial since $\hat{v} \geq 0$. If (b) is true, by substituting the point P in constraint i of $Q^{m,n}$, we get $\sum_{j=1}^t \alpha_j \left\lfloor \beta_i^{(j-1)} / \alpha_j \right\rfloor + \hat{v} \geq \beta_i$, or $\hat{v} \geq \beta_i^{(t)}$, which is true based on (b). \square

Lemma 25. For $i \in M$, $\phi^i(p^{i,t}) = 0$, $t = 1, \dots, n$, and for $i \in K$, $\phi^i(q^{i,n}) = 1$.

Proof. For $i \in M$ and $t = 1, \dots, n$, we have

$$\begin{aligned}
 \phi^i(p^{i,t}) &= \prod_{l=1}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor - \sum_{j=1}^{t-1} \prod_{l=j+1}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor - \prod_{l=t+1}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor \left\lfloor \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rfloor \\
 &= \prod_{l=2}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor \left(\left\lfloor \frac{\beta_i}{\alpha_1} \right\rfloor - \left\lfloor \frac{\beta_i}{\alpha_1} \right\rfloor \right) - \sum_{j=2}^{t-1} \prod_{l=j+1}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor - \prod_{l=t}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor \\
 &= \prod_{l=2}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor - \sum_{j=2}^{t-1} \prod_{l=j+1}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor - \prod_{l=t}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor \\
 &= \prod_{l=3}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor \left(\left\lfloor \frac{\beta_i^{(1)}}{\alpha_2} \right\rfloor - \left\lfloor \frac{\beta_i^{(1)}}{\alpha_2} \right\rfloor \right) - \sum_{j=3}^{t-1} \prod_{l=j+1}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor \\
 &\quad - \prod_{l=t}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor = \dots = \prod_{l=t}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor - \prod_{l=t}^n \left\lfloor \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rfloor = 0.
 \end{aligned}$$

Notice that for $i \in K$ we have $q^{i,n} = p^{i,n} + e_n$, where $e_n = (0, \dots, 0, 1) \in \mathbb{R}^n$. Based on (3.3), it is easy to see that $\phi^i(q^{i,n}) = \phi^i(p^{i,n}) + 1 = 1$. \square

Recall that without loss of generality we have assumed that the set of indices of inequalities used in mixing are $K = \{1, \dots, k\}$, where $\beta_{i-1}^{(n)} \leq \beta_i^{(n)}$, $i = 2, \dots, k$.

Theorem 26. If conditions (3.1) hold, the type I mixed n -step MIR inequality (3.5) defines a facet for $\text{conv}(Q^{m,n})$.

Proof. Consider the support hyperplane of inequality (3.5), i.e.

$$v = \sum_{i=1}^k (\beta_i^{(n)} - \beta_{i-1}^{(n)}) \phi^i(y^i) \quad (3.11)$$

and the face defined by it, i.e. $F_1 = \{(y^1, \dots, y^m, v) \in \text{conv}(Q^{m,n}) : (3.11)\}$. We prove that any generic hyperplane

$$\lambda_0 v + \sum_{i=1}^m \left(\sum_{j=1}^n \lambda_j^i y_j^i \right) = \theta \quad (3.12)$$

that passes through F_1 has to be a scalar multiple of (3.11). For this, consider the point $P^1 = (p^{1,1}, \dots, p^{m,1}, 0) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$. By Lemma 24(a), $P^1 \in Q^{m,n}$ and by Lemma 25, P^1 satisfies (3.11) so $P^1 \in F_1$, and hence must satisfy (3.12) too. That means

$$\sum_{i=1}^m \lambda_1^i \left\lceil \frac{\beta_i}{\alpha_1} \right\rceil = \theta. \quad (3.13)$$

Based on (3.13), hyperplane (3.12) reduces to

$$\lambda_0 v = \sum_{i=1}^m \left(\lambda_1^i \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil - y_1^i \right) - \sum_{j=2}^n \lambda_j^i y_j^i \right). \quad (3.14)$$

For $i \in M$, consider the point $P^{i,2} = (p^{1,1}, \dots, p^{i-1,1}, p^{i,2}, p^{i+1,1}, \dots, p^{m,1}, 0) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$. Again by Lemmas 24 and 25, $P^{i,2} \in F_1$, and hence must satisfy (3.14) too. Substituting $P^{i,2}$, $i \in M$, in (3.14) gives

$$\lambda_1^i = \lambda_2^i \left\lceil \frac{\beta_i^{(1)}}{\alpha_2} \right\rceil, \quad i \in M. \quad (3.15)$$

Based on (3.15), hyperplane (3.14) reduces to

$$\lambda_0 v = \sum_{i=1}^m \left(\lambda_2^i \left(\left\lceil \frac{\beta_i}{\alpha_1} \right\rceil \left\lceil \frac{\beta_i^{(1)}}{\alpha_2} \right\rceil - \left\lceil \frac{\beta_i^{(1)}}{\alpha_2} \right\rceil y_1^i - y_2^i \right) - \sum_{j=3}^n \lambda_j^i y_j^i \right). \quad (3.16)$$

Starting with (3.16), and for each $i \in M$, repeating the same argument using the points $P^{i,3}, P^{i,4}, \dots, P^{i,n} \in F_1$ one after the other, where $P^{i,t} = (p^{1,1}, \dots, p^{i-1,1}, p^{i,t}, p^{i+1,1},$

$\dots, p^{m,1}, 0)$ for $t = 1, \dots, n$, we get the identities

$$\lambda_{t-1}^i = \lambda_t^i \left\lceil \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rceil, \quad t = 2, \dots, n, \quad i \in M. \quad (3.17)$$

Based on (3.17), we get the identities

$$\lambda_t^i = \lambda_n^i \prod_{j=t+1}^n \left\lceil \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rceil, \quad t = 1, \dots, n-1, \quad i \in M, \quad (3.18)$$

which reduce hyperplane (3.16) to

$$\lambda_0 v = \sum_{i=1}^m \lambda_n^i \left(\prod_{l=1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^n \prod_{l=j+1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_j^i \right),$$

or

$$\lambda_0 v = \sum_{i=1}^m \lambda_n^i \phi^i(y^i). \quad (3.19)$$

Now for $i \in K$, consider the point $S^i = (q^{1,n}, \dots, q^{i,n}, p^{i+1,1}, \dots, p^{m,1}, \beta_i^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$. Since $\beta_t^{(n)} \leq \beta_i^{(n)}$ for $t = 1, \dots, i$, by Lemma 24, $S^i \in Q^{m,n}$. By Lemma 25, S^i satisfies (3.11) so $S^i \in F_1$, and hence must satisfy (3.19). Substituting in (3.19) gives

$$\lambda_0 \beta_i^{(n)} = \sum_{t=1}^i \lambda_n^t, \quad i \in K,$$

which implies

$$\lambda_n^i = \lambda_0 \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right), \quad i \in K. \quad (3.20)$$

Identities (3.20) reduce hyperplane (3.19) to

$$\lambda_0 v = \lambda_0 \sum_{i=1}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) + \sum_{i=k+1}^m \lambda_n^i \phi^i(y^i). \quad (3.21)$$

Now for $i = k+1, \dots, m$, consider the point $G^i = (p^{1,1}, \dots, p^{i-1,1}, g^i, p^{i+1,1}, \dots, p^{m,1}, 0) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$, where $g^i \in \mathbb{Z} \times \mathbb{Z}_+^{m-1}$, $\phi^i(g^i) \neq 0$, and g^i has sufficiently large coordinates for $(g^i, 0)$ to satisfy constraint i in $Q^{m,n}$ (clearly such g^i exists because

$\alpha_j > 0, j \in J$). Therefore using Lemma 24, $G^i \in Q^{m,n}$. Also, based on Lemma 25, G^i satisfies (3.11), so $G^i \in F_1$, and hence must satisfy (3.21). Substituting G^i in (3.21), based on Lemma 25 and since $\phi^i(g^i) \neq 0$, we get $\lambda_n^i = 0$. Therefore, $\lambda_n^i = 0, i = k+1, \dots, m$, so (3.21) reduces to $\lambda_0 v = \lambda_0 \sum_{i=1}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i)$, which is λ_0 times (3.11). This completes the proof. \square

Next we address the type II mixed n -step MIR inequality. We will show that the face defined by a type II mixing inequality for $\text{conv}(Q^{m,n})$ has always a dimension of at least $n(m-1)$, and moreover, is a facet if some additional conditions on $(\alpha_1, \dots, \alpha_n)$, β_1 , and β_k are satisfied. To prove this result first we define some more points and establish some properties for them.

Definition 27. Assuming $\left\lfloor \beta_1^{(j-1)} / \alpha_j \right\rfloor \geq 1, j = 2, \dots, n$, define the points $r^t = (r_1^t, \dots, r_n^t) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}, t = 2, \dots, n$, such that

$$r_j^t = \begin{cases} \left\lfloor \beta_1^{(j-1)} / \alpha_j \right\rfloor & \text{for } j = 1, \dots, t-2 \\ \left\lfloor \beta_1^{(j-1)} / \alpha_j \right\rfloor - 1 & \text{for } j = t-1 \\ 2 \left\lfloor \beta_1^{(j-1)} / \alpha_j \right\rfloor + 1 & \text{for } j = t \\ \left\lfloor \beta_1^{(j-1)} / \alpha_j \right\rfloor & \text{for } j = t+1, \dots, n \end{cases}$$

and the point $s = (s_1, \dots, s_n) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$ such that $s = q^{1,n} - e_n$, where $e_n = (0, \dots, 0, 1) \in \mathbb{R}^n$.

Lemma 28. The point $P = (\hat{y}^1, \dots, \hat{y}^m, \hat{v}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$ satisfies constraint 1 of $Q^{m,n}$ if any of the following is true:

- (a). $\hat{y}^1 = r^t$ for some $t \in \{2, \dots, n\}$ and $\hat{v} \geq \beta_1^{(n)} + \alpha_{t-1} - \alpha_t \left\lceil \beta_1^{(t-1)} / \alpha_t \right\rceil$,
- (b). $\hat{y}^1 = s$ and $\hat{v} \geq \alpha_n + \beta_1^{(n)}$.

Proof. If (a) is true, by substituting the point P in constraint 1 of $Q^{m,n}$, we get

$$\sum_{j=1}^{t-1} \alpha_j \left\lfloor \beta_1^{(j-1)} / \alpha_j \right\rfloor - \alpha_{t-1} + \alpha_t \left(2 \left\lfloor \beta_1^{(t-1)} / \alpha_t \right\rfloor + 1 \right) + \sum_{j=t+1}^n \alpha_j \left\lfloor \beta_1^{(j-1)} / \alpha_j \right\rfloor + \hat{v} \geq \beta_1.$$

This simplifies to $\hat{v} \geq \beta_1^{(n)} + \alpha_{t-1} - \alpha_t \left\lfloor \beta_1^{(t-1)} / \alpha_t \right\rfloor$, which is true by (a). If (b) is true, by substituting the point P in constraint 1 of $Q^{m,n}$, we get $\sum_{j=1}^n \alpha_j \left\lfloor \beta_1^{(j-1)} / \alpha_j \right\rfloor - \alpha_n + \hat{v} \geq \beta_1$, or $\hat{v} \geq \alpha_n + \beta_1^{(n)}$, which is true by (b). \square

Lemma 29. $\phi^1(r^t) = 1$ for $t = 2, \dots, n$, and $\phi^1(s) = 2$.

Proof. The function $\phi^1(y^1)$ can be written as

$$\begin{aligned} \phi^1(y^1) &= \prod_{l=1}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor - \sum_{j=1}^n \prod_{l=j+1}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor y_j^1 \\ &= \prod_{l=2}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor + \prod_{l=2}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor \left\lfloor \frac{\beta_1}{\alpha_1} \right\rfloor - \sum_{j=1}^n \prod_{l=j+1}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor y_j^1 \\ &= \prod_{l=2}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor + \prod_{l=2}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor \left(\left\lfloor \frac{\beta_1}{\alpha_1} \right\rfloor - y_1^1 \right) - \sum_{j=2}^n \prod_{l=j+1}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor y_j^1 \\ &= \prod_{l=3}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor + \sum_{j=1}^2 \prod_{l=j+1}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor \left(\left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor - y_j^1 \right) - \\ &\quad \sum_{j=3}^n \prod_{l=j+1}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor y_j^1 = \dots = 1 + \sum_{j=1}^n \prod_{l=j+1}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor \left(\left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor - y_j^1 \right). \end{aligned} \tag{3.22}$$

Based on (3.22), for $t = 2, \dots, n$ we have

$$\begin{aligned} \phi^1(r^t) &= 1 + \prod_{l=t}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor + \prod_{l=t+1}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor \left(\left\lfloor \frac{\beta_1^{(t-1)}}{\alpha_t} \right\rfloor - 2 \left\lfloor \frac{\beta_1^{(t-1)}}{\alpha_t} \right\rfloor - 1 \right) \\ &= 1 + \prod_{l=t}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor - \prod_{l=t}^n \left\lfloor \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rfloor = 1, \end{aligned}$$

and

$$\phi^1(s) = 1 + \left(\left\lfloor \frac{\beta_1^{(n-1)}}{\alpha_n} \right\rfloor - \left\lfloor \frac{\beta_1^{(n-1)}}{\alpha_n} \right\rfloor + 1 \right) = 2.$$

□

Theorem 30. *If conditions (3.1) hold, the type II mixed n -step MIR inequality defines a face of dimension at least $n(m-1)$ for $\text{conv}(Q^{m,n})$. Moreover, this inequality defines a facet for $\text{conv}(Q^{m,n})$ if the following additional conditions are satisfied:*

- (a). $\left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor \geq 1, j = 2, \dots, n,$
- (b). $\beta_k^{(n)} - \beta_1^{(n)} \geq \max \left\{ \alpha_{j-1} - \alpha_j \left\lfloor \beta_1^{(j-1)}/\alpha_j \right\rfloor, j = 2, \dots, n \right\}.$

Proof. Consider the support hyperplane of inequality (3.6), i.e.

$$v = \sum_{i=1}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) + \left(\alpha_n - \beta_k^{(n)} \right) (\phi^1(y^1) - 1), \quad (3.23)$$

and the face defined by it, i.e. $F_2 = \{(y^1, \dots, y^m, v) \in \text{conv}(Q^{m,n}) : (3.23)\}$. We prove that any generic hyperplane defined by $(\lambda^1, \dots, \lambda^m, \lambda_0, \theta) \in \mathbb{R}^{mn+2}$, i.e.

$$\lambda_0 v + \sum_{i=1}^m \left(\sum_{j=1}^n \lambda_j^i y_j^i \right) = \theta, \quad (3.24)$$

that passes through F_2 is the linear combination of at most $n+1$ linearly independent hyperplanes, making F_2 a face of dimension at least $mn+1-(n+1) = n(m-1)$.

Consider the point $S^1 = (q^{1,n}, p^{2,1}, \dots, p^{m,1}, \beta_1^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$. As argued in the proof of Theorem 26, $S^1 \in Q^{m,n}$. Moreover, using Lemma 25, it is easy to verify that S^1 satisfies (3.23). So $S^1 \in F_2$ and hence must satisfy (3.24). Substituting into (3.24) gives

$$\lambda_0 \beta_1^{(n)} + \sum_{j=1}^n \lambda_j^1 \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor + \sum_{i=2}^m \lambda_1^i \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor = \theta. \quad (3.25)$$

Based on (3.25), hyperplane (3.24) reduces to

$$\lambda_0 \left(v - \beta_1^{(n)} \right) + \sum_{j=1}^n \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor \right) = \sum_{i=2}^m \left(\lambda_1^i \left(\left\lfloor \frac{\beta_i}{\alpha_1} \right\rfloor - y_1^i \right) - \sum_{j=2}^n \lambda_j^i y_j^i \right). \quad (3.26)$$

Consider the points $R^{i,t} = (q^{1,n}, p^{2,1}, \dots, p^{i-1,1}, p^{i,t}, p^{i+1,1}, \dots, p^{m,1}, \beta_1^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$, $i = 2, \dots, m$, $t = 2, \dots, n$. By Lemma 24, these points belong to $Q^{m,n}$, and by Lemma 25, they satisfy (3.23). Therefore $R^{i,t} \in F_2$, $i = 2, \dots, m$, $t = 2, \dots, n$. Starting with hyperplane (3.26), and for each $i \in \{2, \dots, m\}$, substituting the points $R^{i,2}, \dots, R^{i,n}$ in the hyperplane, one after the other, we get

$$\lambda_{t-1}^i = \lambda_t^i \left\lfloor \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rfloor, \quad t = 2, \dots, n, \quad i = 2, \dots, m. \quad (3.27)$$

From (3.27) we get

$$\lambda_t^i = \lambda_n^i \prod_{j=t+1}^n \left\lfloor \frac{\beta_i^{(j-1)}}{\alpha_j} \right\rfloor, \quad t = 2, \dots, n, \quad i = 2, \dots, m, \quad (3.28)$$

which reduces (3.26) to

$$\lambda_0 \left(v - \beta_1^{(n)} \right) + \sum_{j=1}^n \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor \right) = \sum_{i=2}^m \lambda_n^i \phi^i(y^i). \quad (3.29)$$

Now consider the points $S^i = (q^{1,n}, \dots, q^{i,n}, p^{i+1,1}, \dots, p^{m,1}, \beta_i^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$, $i = 2, \dots, k$, that were used in the proof of Theorem 26. We argued that these points belong to $Q^{m,n}$. Moreover, using Lemma 25, it can be easily verified that they satisfy (3.23), so $S^i \in F_2$, $i = 2, \dots, k$. Therefore, they must satisfy (3.29). Substituting S^i , $i = 2, \dots, k$, in (3.29), we get

$$\lambda_0 \left(\beta_i^{(n)} - \beta_1^{(n)} \right) = \sum_{t=2}^i \lambda_n^t, \quad i = 2, \dots, k,$$

which implies

$$\lambda_n^i = \lambda_0 \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right), \quad i = 2, \dots, k. \quad (3.30)$$

Identities (3.30) reduce hyperplane (3.29) to

$$\lambda_0 \left(v - \sum_{i=2}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) - \beta_1^{(n)} \right) + \sum_{j=1}^n \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor \right) = \sum_{i=k+1}^m \lambda_n^i \phi^i(y^i). \quad (3.31)$$

Now for $i = k+1, \dots, m$, consider the points $H^i = (q^{1,n}, p^{2,1}, \dots, p^{i-1,1}, h^i, p^{i+1,1}, \dots, p^{m,1}, \beta_1^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$, where $h^i \in \mathbb{Z} \times \mathbb{Z}_+^{m-1}$, $\phi^i(h^i) \neq 0$, and h^i has sufficiently large coordinates for $(h^i, \beta_1^{(n)})$ to satisfy constraint i in $Q^{m,n}$ (clearly such h^i exists because $\alpha_j > 0, j \in J$). Therefore using Lemma 24, $H^i \in Q^{m,n}$. Also, based on Lemma 25, H^i satisfies (3.23), so $H^i \in F_2$, and hence must satisfy (3.31). Substituting H^i in (3.31), based on Lemma 25 and since $\phi^i(h^i) \neq 0$, we get $\lambda_n^i = 0$. Therefore, $\lambda_n^i = 0, i = k+1, \dots, m$, so (3.31) reduces to

$$\lambda_0 \left(v - \sum_{i=2}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) - \beta_1^{(n)} \right) + \sum_{j=1}^n \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor \right) = 0. \quad (3.32)$$

So we have shown that in the generic hyperplane (3.24) defined by $(\lambda^1, \dots, \lambda^m, \lambda_0, \theta) \in \mathbb{R}^{mn+2}$, at most $(\lambda^1, \lambda_0) \in \mathbb{R}^{n+1}$ are independent. That means the generic hyperplane can be the linear combination of at most $n+1$ linearly independent hyperplanes. This proves that F_2 is a face of dimension at least $n(m-1)$.

To prove the second part of the theorem, assume the additional conditions (a) and (b) are satisfied. Notice that (3.23) can also be written as

$$v - \sum_{i=2}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) - \beta_1^{(n)} = \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)} \right) (\phi^1(y^1) - 1). \quad (3.33)$$

Any point on F_2 satisfies both (3.32) and (3.33). These two identities together imply that the identity

$$\lambda_0 \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)} \right) (\phi^1(y^1) - 1) + \sum_{j=1}^n \lambda_j^1 \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor \right) = 0 \quad (3.34)$$

holds for any point on F_2 . Replacing for $\phi^1(y^1)$ from (3.22), identity (3.34) can be written as

$$\sum_{j=1}^n c_j \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor \right) = 0 \quad (3.35)$$

where $c_j = \lambda_j^1 - \lambda_0 \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)} \right) \prod_{l=j+1}^n \left\lfloor \beta_1^{(l-1)} / \alpha_l \right\rfloor$. Now, consider the point $U = (s, q^{2,n}, \dots, q^{k,n}, p^{k+1,1}, \dots, p^{m,1}, \alpha_n + \beta_1^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$ (condition (a) guarantees that $s \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$). By Lemma 28(b), U satisfies constraint 1 of $Q^{m,n}$, and by Lemma 24, it satisfies constraints $2, \dots, m$ of $Q^{m,n}$, therefore $U \in Q^{m,n}$. Also using Lemmas 25 and 29, it is easy to verify that U lies on (3.23). Therefore $U \in F_2$ and must satisfy (3.35). Similarly, for $t = 2, \dots, n$ consider the point $V^t = (r^t, q^{2,n}, \dots, q^{k,n}, p^{k+1,1}, \dots, p^{m,1}, \beta_k^{(n)}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+$ (condition (a) guarantees that $r^t \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$). By Lemma 28 and condition (b) of this theorem, V^t satisfies the first constraint of $Q^{m,n}$, and by Lemma 24, it satisfies constraints $2, \dots, m$ of $Q^{m,n}$. Therefore $V^t \in Q^{m,n}, t = 2, \dots, n$. Moreover, using Lemmas 25 and 29, it can be easily verified that the points $V^t, t = 2, \dots, n$, lie on hyperplane (3.23) and so $V^t \in F_2, t = 2, \dots, n$, and must satisfy (3.35). Starting with identity (3.35), and substituting in it the points $U, V^n, V^{n-1}, \dots, V^2$ one by one in that order, we get $c_n = 0, c_{n-1} = 0, \dots, c_1 = 0$, respectively. Therefore

$$\lambda_j^1 = \lambda_0 \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)} \right) \prod_{l=j+1}^n \left\lfloor \beta_1^{(l-1)} / \alpha_l \right\rfloor, j = 1, \dots, n. \quad (3.36)$$

Identities (3.36) reduce hyperplane (3.32) to

$$\begin{aligned} & \lambda_0 \left(v - \sum_{i=2}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) - \beta_1^{(n)} \right. \\ & \left. + \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)} \right) \sum_{j=1}^n \prod_{l=j+1}^n \left\lceil \frac{\beta_1^{(l-1)}}{\alpha_l} \right\rceil \left(y_j^1 - \left\lfloor \frac{\beta_1^{(j-1)}}{\alpha_j} \right\rfloor \right) \right) = 0. \end{aligned} \quad (3.37)$$

Using (3.22), hyperplane (3.37) can be written as

$$\begin{aligned} & \lambda_0 \left(v - \sum_{i=2}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) - \beta_1^{(n)} - \left(\alpha_n + \beta_1^{(n)} - \beta_k^{(n)} \right) (\phi^1(y^1) - 1) \right) = 0, \\ & \text{or } \lambda_0 \left(v - \sum_{i=1}^k \left(\beta_i^{(n)} - \beta_{i-1}^{(n)} \right) \phi^i(y^i) - \left(\alpha_n - \beta_k^{(n)} \right) (\phi^1(y^1) - 1) \right) = 0, \end{aligned}$$

which is simply λ_0 times (3.23). This proves that F_2 defines a facet for $\text{conv}(Q^{m,n})$. \square

Example 1. Consider the 3-mixing set with 2 rows $Q^{2,3} = \{(y^1, y^2, v) \in (\mathbb{Z} \times \mathbb{Z}_+^2)^2 \times \mathbb{R}_+ : 31y_1^1 + 10y_2^1 + 3y_3^1 + v \geq 89; 31y_1^2 + 10y_2^2 + 3y_3^2 + v \geq 59\}$. Therefore $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (31, 10, 3)$, $\beta_1 = 89$, $\beta_2 = 59$, and we have $\beta_1^{(1)} = 27$, $\beta_1^{(2)} = 7$, $\beta_1^{(3)} = 1$, $\beta_2^{(1)} = 28$, $\beta_2^{(2)} = 8$, and $\beta_2^{(3)} = 2$. So $\lceil \beta_1^{(1)}/\alpha_2 \rceil = \lceil \beta_1^{(2)}/\alpha_3 \rceil = \lceil \beta_2^{(1)}/\alpha_2 \rceil = \lceil \beta_2^{(2)}/\alpha_3 \rceil = 3$ and it is easily verified that conditions (3.1) are satisfied. Therefore, based on (3.5) and (3.6), the type I and type II mixed 3-step MIR inequalities obtained from the two defining inequalities of $Q^{2,3}$ are as follows (note that $\beta_1^{(3)} \leq \beta_2^{(3)}$):

$$v \geq (27 - 9y_1^1 - 3y_2^1 - y_3^1) + (18 - 9y_1^2 - 3y_2^2 - y_3^2), \quad (3.38)$$

$$v \geq (27 - 9y_1^1 - 3y_2^1 - y_3^1) + (18 - 9y_1^2 - 3y_2^2 - y_3^2) + (27 - 9y_1^1 - 3y_2^1 - y_3^1 - 1). \quad (3.39)$$

Based on Theorem 26, inequality (3.38) defines a facet for $\text{conv}(Q^{2,3})$. The additional conditions (a) and (b) of Theorem 30 are also satisfied, i.e.

- (a) $\left\lfloor \beta_1^{(1)}/\alpha_2 \right\rfloor = \left\lfloor \beta_2^{(1)}/\alpha_3 \right\rfloor = 2 \geq 1$, and
 (b) $\beta_2^{(3)} - \beta_1^{(3)} = 1 \geq 1 = \max \left\{ \alpha_1 - \alpha_2 \left\lceil \beta_1^{(1)}/\alpha_2 \right\rceil, \alpha_2 - \alpha_3 \left\lceil \beta_1^{(2)}/\alpha_3 \right\rceil \right\}$.

Therefore, based on Theorem 30, inequality (3.39) also defines a facet for $\text{conv}(Q^{2,3})$.

Similarly, consider the 2-mixing set $Q^{2,2} = \{(y^1, y^2, v) \in (\mathbb{Z} \times \mathbb{Z}_+)^2 \times \mathbb{R}_+ : 31y_1^1 + 10y_2^1 + v \geq 89; 31y_1^2 + 10y_2^2 + v \geq 59\}$. It is easy to see that conditions (3.1) as well as conditions (a) and (b) of Theorem 30 are satisfied as $\alpha_1, \alpha_2, \beta_1$, and β_2 have the same values as above. Therefore, the type I and type II mixed 2-step MIR inequalities

$$v \geq 7(9 - 3y_1^1 - y_2^1) + (6 - 3y_1^2 - y_2^2)$$

$$v \geq 7(9 - 3y_1^1 - y_2^1) + (6 - 3y_1^2 - y_2^2) + 2(9 - 3y_1^1 - y_2^1 - 1)$$

are facet-defining for $\text{conv}(Q^{2,2})$ based on Theorems 26 and 30, respectively. \square

III.3 Mixed n -step MIR Inequalities for General MIP

As mentioned in Section II.3, n -step MIR can be used to generate valid inequalities for the general single-constraint mixed integer knapsack set Y_1 [78]. In this section, we show that the mixed n -step MIR inequality for the set $Q^{m,n}$ can be used to generate mixed n -step MIR inequalities for the general multi-constraint mixed integer set Y_m . This implies that mixed n -step MIR can generate valid inequalities based on multiple constraints for a general MIP because the feasible set of a general MIP with m constraints can be relaxed to a set of the form Y_m as follows: Define the feasible set of a general MIP as $\{(x, w) \in \mathbb{Z}_+^N \times \mathbb{R}_+^{|C|} : \sum_{t \in T} a_{it}x_t + \sum_{t \in C} c_{it}w_t = b_i, i = 1, \dots, m\}$, where C is the index set of the continuous variables w , and $b_i, a_{it}, c_{it} \in \mathbb{R}$ for all i and j . This set can be relaxed to $\{(x, w) \in \mathbb{Z}_+^N \times \mathbb{R}_+^{|C|} : \sum_{t \in T} a_{it}x_t + \sum_{t \in C: c_{it} > 0} c_{it}w_t \geq b_i, i = 1, \dots, m\}$. Representing $\sum_{t \in C: c_{it} > 0} c_{it}w_t$ by s_i , we get the set Y_m .

Any subset of the m rows in Y_m can be used to generate a mixed n -step MIR

inequality for this set. Like before without loss of generality, we assume this subset of rows is $K = \{1, \dots, k\}$, where $k \leq m$. A set of n parameters must be chosen to generate the mixed n -step MIR inequality. We denote the vector of these parameters by $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\alpha \in \mathbb{R}^n$ and $\alpha > 0$. As we will see, these parameters must satisfy the n -step MIR conditions for all rows in K , i.e.

$$\alpha_j \left\lceil b_i^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}, j = 1, \dots, n, i \in K \quad (3.40)$$

(like before we also assume $b_i^{(j-1)} / \alpha_j \notin \mathbb{Z}$, $j = 1, \dots, n$, $i \in K$, to avoid trivial inequalities). Notice that conditions (3.40) are on the parameters α_j chosen by the user and no conditions on coefficients a_{it} in Y_m are required. Without loss of generality, we also assume the rows are indexed such that $b_{i-1}^{(n)} \leq b_i^{(n)}$, $i = 2, \dots, k$. Here we present the type I mixed n -step MIR inequality for Y_m . The type II can be generated in a similar fashion.

Let $a_t = (a_{1t}, a_{2t}, \dots, a_{kt})$ and $b = (b_1, \dots, b_k)$ and let $\pi : \mathbb{R}^k \rightarrow \{0, \dots, n\}^k$ be a mapping. For $i \in K$ and $p = 0, \dots, n$, let $T_p^i := \{t \in T : \pi(a_t)_i = p\}$, where $\pi(a_t)_i$ is the i th component of $\pi(a_t)$.

Definition 31. *The mixed n -step MIR function $\sigma_{\alpha,b}^n : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as follows*

$$\sigma_{\alpha,b}^n(d) = \min_{\bar{\pi} \in \{0, \dots, n\}^k} \left\{ \sum_{i=1}^k \left(b_i^{(n)} - b_{i-1}^{(n)} \right) \delta_{\alpha,b_i}^\pi(d_i) + u^\pi(d) : \pi(d) = \bar{\pi} \right\}, \quad (3.41)$$

where

$$\delta_{\alpha,b_i}^\pi(d) = \begin{cases} \sum_{j=1}^p \prod_{l=j+1}^n \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{d_i^{(j-1)}}{\alpha_j} \right\rfloor + \prod_{l=p+2}^n \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{d_i^{(p)}}{\alpha_{p+1}} \right\rfloor, & \pi(d)_i = p; \\ \sum_{j=1}^n \prod_{l=j+1}^n \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{d_i^{(j-1)}}{\alpha_j} \right\rfloor & \pi(d)_i = n \end{cases}$$

and

$$u^\pi(d) := \max\{0, d_i^{(n)} \text{ for all } i \text{ that } \pi(d)_i = n\}.$$

Theorem 32. *Given a positive parameter vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ which satisfies conditions (3.40), the mixed n -step MIR inequality*

$$\sum_{t \in T} \sigma_{\alpha,b}^n(a_t)x_t + \bar{s} \geq \sigma_{\alpha,b}^n(b) \quad (3.42)$$

is valid for Y_m , where $\bar{s} \in \mathbb{R}_+$ is a variable such that $\bar{s} \geq s_i$ for all $i \in K$.

Proof. Given a mapping π , each constraint of Y_m can be relaxed in the same way that the defining constraint of Y_1 is relaxed in [78]. In other words, for $i \in K$, constraint i of Y_m can be relaxed to

$$\sum_{p=0}^{n-1} \sum_{t \in T_p^i} \left(\sum_{j=1}^p \alpha_j \left\lfloor \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rfloor + \alpha_{p+1} \left\lceil \frac{a_{it}^{(p)}}{\alpha_{p+1}} \right\rceil \right) x_t + \sum_{t \in T_n^i} \left(\sum_{j=1}^n \alpha_j \left\lfloor \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rfloor + a_{it}^{(n)} \right) x_t \geq b_i. \quad (3.43)$$

Notice that this is a relaxation because for any $p \in \{0, 1, \dots, n\}$

$$a_{it} = \sum_{j=1}^p \alpha_j \left\lfloor a_{it}^{(j-1)} / \alpha_j \right\rfloor + a_{it}^{(p)} \quad (3.44)$$

and so

$$a_{it} \leq \sum_{j=1}^p \alpha_j \left\lfloor a_{it}^{(j-1)} / \alpha_j \right\rfloor + \alpha_{p+1} \left\lceil a_{it}^{(p)} / \alpha_{p+1} \right\rceil. \quad (3.45)$$

In other words, to get (3.43), the coefficient a_{it} in every row $i \in K$ of Y_m is relaxed to the right-hand side of (3.45) for $t \in T_p^i$, $p = 0, 1, \dots, n-1$, and is replaced with the right-hand side of identity (3.44) for $t \in T_n^i$. Rearranging the terms of (3.43), we get

$$\sum_{j=1}^n \alpha_j \left(\sum_{t \in T_{j-1}^i} \left\lfloor \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rfloor x_t + \sum_{p=j}^n \sum_{t \in T_p^i} \left\lfloor \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rfloor x_t \right) + \sum_{t \in T_n^i} a_{it}^{(n)} x_t + s_i \geq b_i, i \in K. \quad (3.46)$$

Now for $i \in K$ and $j = 1, \dots, n$, the expression $\sum_{t \in T_{j-1}^i} \left\lfloor a_{it}^{(j-1)} / \alpha_j \right\rfloor x_t + \sum_{p=j}^n \sum_{t \in T_p^i} \left\lfloor a_{it}^{(j-1)} / \alpha_j \right\rfloor x_t$ in (3.46) is an integer (note that for $j = 2, \dots, n$ it is also nonnegative) and can be treated as y_j^i in $\widehat{Q}^{m,n}$. Also for $i \in K$, the expression $\sum_{t \in T_n^i} a_{it}^{(n)} x_t + s_i$ is

nonnegative and can be treated as v_i in $\widehat{Q}^{m,n}$. We choose the upper bound variable \bar{v} in (3.9) to be $\sum_{t \in T} u^\pi(a_t)x_t + \bar{s}$. Since by assumption conditions (3.40) hold, according to Corollary 22, the type I mixed n -step MIR inequality for $\widehat{Q}^{m,n}$ (inequality (3.9)), when y_j^i and \bar{v} are replaced with their aforementioned corresponding expressions, is valid for Y_m . That is

$$\sum_{t \in T} u^\pi(a_t)x_t + \bar{s} \geq \sum_{i=1}^k \left(b_i^{(n)} - b_{i-1}^{(n)} \right) \left(\prod_{l=1}^n \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{j=1}^n \prod_{l=j+1}^n \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\sum_{t \in T_{j-1}^i} \left\lceil \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rceil x_t + \sum_{p=j}^n \sum_{t \in T_p^i} \left\lceil \frac{a_{it}^{(j-1)}}{\alpha_j} \right\rceil x_t \right) \right). \quad (3.47)$$

Putting all multiples of x_t in (3.47) together for each $t \in T$, we can write it as

$$\sum_{t \in T} \left(\sum_{i=1}^k \left(b_i^{(n)} - b_{i-1}^{(n)} \right) \delta_{\alpha, b_i}^\pi(a_j) + u^\pi(a_t) \right) x_t + \bar{s} \geq \sum_{i=1}^n \left(b_i^{(n)} - b_{i-1}^{(n)} \right) \prod_{l=1}^n \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil. \quad (3.48)$$

We would like to choose $\pi(a_t)$ such that we get the strongest inequality, i.e. such that the coefficient of x_t in (3.48) is minimized. Therefore the the smallest coefficient for x_t will be obtained by $\sigma_{\alpha, b}^n(a_t)$. Also, $\sigma_{\alpha, b}^n(b) = \sum_{i=1}^n \left(b_i^{(n)} - b_{i-1}^{(n)} \right) \prod_{l=1}^n \left\lceil b_i^{(l-1)} / \alpha_l \right\rceil$ as it can be easily verified that the minimum in (3.41) in case of $\sigma_{\alpha, b}^n(b)$ is achieved at any $\bar{\pi}$, where $\bar{\pi}_i \neq n$ for all $i \in K$. Therefore (3.48) reduces to (3.42) and the proof is complete. \square

Notice that one possible choice for \bar{s} that guarantees $\bar{s} > s_i$ for all $i \in K$ is $\bar{s} = \sum_{i=1}^k s_i$. Theorem 32 shows that a mixed n -step MIR inequality for k constraints can be simply obtained by applying the corresponding mixed n -step MIR function $\sigma_{\alpha, b}^n$ on the coefficient vectors of the variables and the right-hand side vector. Figure 1 shows an example of the function $\sigma_{\alpha, b}^2(d_1, d_2)$ with $\alpha = (\alpha_1, \alpha_2) = (25, 10)$ and $b = (b_1, b_2) = (39, 18)$ for $(d_1, d_2) \in [-25, 25]^2$. As we see in Theorem 32, conditions (3.40) are only on the parameters α_j chosen by the user and no conditions on coefficients

a_{it} in Y_m are required. An interesting question is whether it is always possible to find a positive parameter vector $\alpha \in \mathbb{R}^n$ such that it satisfies conditions (3.40). The answer is yes. Given the set of rows in K with the right-hand sides b_1, \dots, b_k , there is an infinite number of choices for the parameter vector α that satisfy conditions (3.40). For $i \in K$, $j = 2, \dots, n$, and $l \in \mathbb{N}$, define the intervals $I_i^{j,l}$ in \mathbb{R}_+ as follows:

$$I_i^{j,l} = \begin{cases} \left(\frac{b_i^{(j-1)}}{l}, \frac{\alpha_{j-1}}{l} \right] & \text{for } 2 \leq l < \tau_i^j, \\ \left(\frac{b_i^{(j-1)}}{l}, \frac{b_i^{(j-1)}}{l-1} \right) & \text{for } l \geq \tau_i^j. \end{cases}$$

where $\tau_i^j = \left\lceil \alpha_{j-1} / (\alpha_{j-1} - b_i^{(j-1)}) \right\rceil$. Then one can choose the elements of the parameter vector α in a recursive fashion as follows:

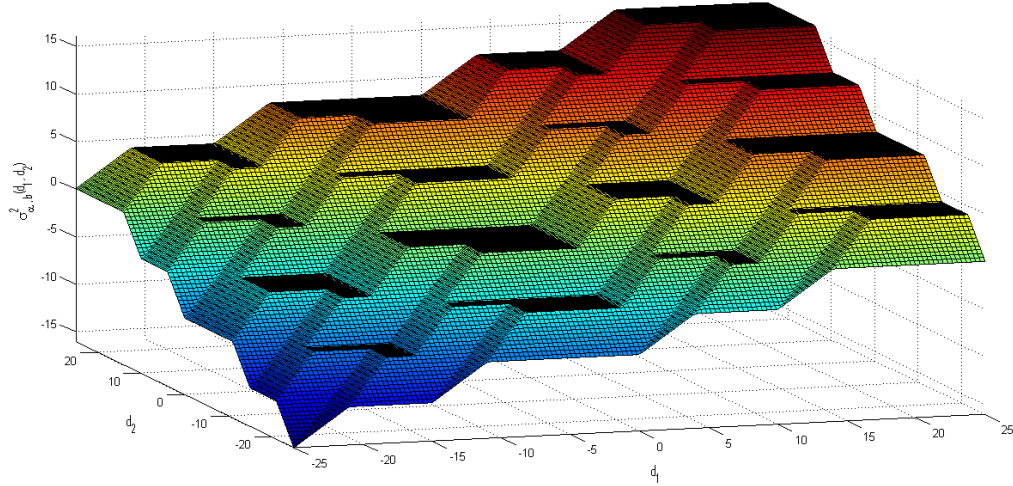


Fig. 1. $\sigma_{\alpha,b}^2(d_1, d_2)$ over $[-25, 25]^2$ with $\alpha = (25, 10)$ and $b = (39, 18)$

Step 1. Pick a positive value for α_1 ;

Step 2. For $j := 2, \dots, n$ do

Pick a value for α_j such that $\alpha_j \in \cap_{i \in K} \cup_{l=2}^{+\infty} I_i^{j,l}$;

We see that in iteration j of Step 2, the set of possible values for α_j depends on the values picked for $\alpha_1, \dots, \alpha_{j-1}$. Notice that for any i, j and l , we have $\lceil b_i^{(j-1)}/\alpha_j \rceil = l$ if $\alpha_j \in I_i^{j,l}$. Based on the definitions of τ_i^j and the intervals $I_i^{j,l}$, it can be easily verified that each α_j picked from the set in Step 2 satisfies the conditions $\alpha_j \lceil b_i^{(j-1)}/\alpha_j \rceil \leq \alpha_{j-1}$ for $i \in K$. Moreover, observe that for each $j \in \{2, \dots, n\}$, the set $\cap_{i \in K} \cup_{l=2}^{+\infty} I_i^{j,l}$ contains the interval $(0, \min\{b_i^{(j-1)}/(\tau_i^j - 1), i \in K\})$ except for the discrete values $b_i^{(j-1)}/l$, $l \in \mathbb{N}, l \geq \tau_i^j$. Therefore there are always infinitely many choices for each α_j . We note that the intervals presented in [45] for the 2-step MIR inequality are the special case of $I_i^{j,l}$ for $n = 2$, $k = 1$, and $\alpha_1 = 1$.

III.4 Mixed n -step MIR Inequalities for Special Structures

The capacitated lot-sizing problem [104, 117, 118] and the capacitated facility location problem [1, 2, 117] have been studied for years. In this section, we introduce useful generalizations of these two problems, which we refer to as the multi-module lot-sizing problem (MML) and the multi-module facility location problem (MMF), respectively, and show that the mixed n -step MIR inequalities can be used to generate valid inequalities for them. The mixed n -step MIR inequalities for MML generalize the (k, l, S, I) inequalities for the constant-capacity lot-sizing problem (CCL) [70, 104] and the mixed n -step MIR inequalities for MMF generalize the mixed MIR inequalities for the constant-capacity facility location problem (CCF) [1, 2, 70].

III.4.1 Multi-Module Lot-Sizing (MML)

We first define the multi-module lot-sizing problem (MML). Let $T := \{1, \dots, m\}$ be the set of time periods and $\{\alpha_1, \dots, \alpha_n\}$ be the set of capacities of n available capacity modules. In each period the total capacity can be the summation of some integer

multiples of $\alpha_1, \dots, \alpha_n$. In MML the goal is to find a production plan that minimizes the sum of production, inventory, and module setup costs over all periods while meeting the demands (without backlogging) and satisfying capacity constraints. Let x_t be the production, s_t be the inventory at the end of period t , and z_t^j be the number of modules of capacity α_j , $j = 1, \dots, n$, used in period t . Then MML is $\min\{\sum_{t \in T} p_t x_t + \sum_{t \in T} h_t s_t + \sum_{t \in T} \sum_{j=1}^n f_t^j z_t^j : (x, s, z) \in X^{MML}\}$, where

$$X^{MML} = \left\{ (x, s, z) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{Z}_+^{m \times n} : \right.$$

$$s_{t-1} + x_t = d_t + s_t, \quad t \in T \quad (3.49)$$

$$x_t \leq \sum_{j=1}^n \alpha_j z_t^j, \quad t \in T \quad \left. \right\}, \quad (3.50)$$

and d_t , p_t , h_t , and f_t^j are the demand, production cost per unit, inventory cost per unit, and the setup cost per module of capacity α_j , $j = 1, \dots, n$, in period t , respectively, and $s_0 = 0$.

When $\alpha_1 = \alpha_2 = \dots = \alpha_n = C$, the capacity constraints (3.50) simplify to $x_t \leq C y_t$, $t \in T$, where $y_t = \sum_{j=1}^n z_t^j$ (variables z_t^j are not needed anymore), and MML reduces to CCL, the constant capacity lot sizing problem in which capacity in each time period is a multiple of C . The special case of CCL in which $y_t \in \{0, 1\}$, $t \in T$ was discussed in Section II.4.

Here we show that the mixed n -step MIR can be used to get valid inequalities for X^{MML} . These inequalities generalize the (k, l, S, I) inequalities for X^{CCL} to the case of multiple capacity modules. First, we construct the *base inequalities* for which the mixed n -step MIR inequalities will be written. we follow the notation used in Section .

For any $k, l \in T$, where $k < l$, let $S \subseteq \{k, \dots, l\}$. For $i \in S$, let $S_i = S \cap \{k, \dots, i\}$ and $b_i = \sum_{t=k}^{n_i-1} d_t$, where n_i is defined as in (2.17). Adding up equalities (3.49) from

period k to period $n_i - 1$, we get

$$s_{k-1} + \sum_{t=k}^{n_i-1} x_t = b_i + s_{n_i-1}. \quad (3.51)$$

Note that $S_i \subseteq \{k, \dots, n_i - 1\}$ by definition, and that this aggregation is similar to the one performed for *CCL* in Section II.4. If we relax x_t , $t \in S_i$, in (3.51) to its upper bound based on (3.50) and drop $s_{n_i-1} (\geq 0)$, we get the following valid inequality:

$$s_{k-1} + \sum_{t \in \{k, \dots, n_i-1\} \setminus S_i} x_t + \sum_{t \in S_i} \sum_{j=1}^n \alpha_j z_t^j \geq b_i. \quad (3.52)$$

Setting $v_i := s_{k-1} + \sum_{t \in \{k, \dots, n_i-1\} \setminus S_i} x_t$ and $y_j^i := \sum_{t \in S_i} z_t^j$, $j = 1, \dots, n$, inequality (3.52) becomes

$$\sum_{j=1}^n \alpha_j y_j^i + v_i \geq b_i, \quad (3.53)$$

which is of the same form as the defining inequalities of $Q^{m,n}$ (notice that $v_i \in \mathbb{R}_+$, $y_j^i \in \mathbb{Z}_+$, $j = 1, \dots, n$). Let $I \subseteq S$. We get an inequality like (3.53) for each $i \in I$. Without loss of generality and for simplicity of notation assume the parameter vector for mixed n -step MIR is $\alpha = (\alpha_1, \dots, \alpha_n)$ and also $I = \{1, \dots, |I|\}$ such that $b_{i-1}^{(n)} \leq b_i^{(n)}$, $i \in I$. Now if $\alpha_j \left[b_i^{(j-1)} / \alpha_j \right] \leq \alpha_{j-1}$, $j = 2, \dots, n$, $i \in I$, then by letting $\bar{v} = s_{k-1} + \sum_{t \in \{k, \dots, n_{|I|-1}\} \setminus S} x_t$ (note that $\bar{v} \geq v_i$ for all $i \in I$), based on Corollary 22, the mixed n -step MIR inequalities

$$s_{k-1} + \sum_{t \in \{k, \dots, n_{|I|-1}\} \setminus S} x_t \geq \sum_{i=1}^{|I|} \left(b_i^{(n)} - b_{i-1}^{(n)} \right) \phi^i(y^i), \quad (3.54)$$

$$s_{k-1} + \sum_{t \in \{k, \dots, n_{|I|-1}\} \setminus S} x_t \geq \sum_{i=1}^{|I|} \left(b_i^{(n)} - b_{i-1}^{(n)} \right) \phi^i(y^i) + \left(\alpha_n - b_{|I|}^{(n)} \right) \left(\phi_n^1(y^1) - 1 \right) \quad (3.55)$$

are valid for X^{MML} , where $y_j^i = \sum_{t \in S_i} z_t^j$. We refer to inequalities (3.54) and (3.55) as the *type I and type II multi-module (k, l, S, I) inequalities*. The (k, l, S, I) inequalities for X^{CCL} presented in [70, 104] are the special case of (3.54) for $n = 1$ (the constant

capacity case).

Remark 2. A special case of MML is when in each period t only modules of a specific capacity C_t are available but the capacity of modules in different periods are not necessarily the same. This is the well-known capacitated lot-sizing problem (CL) [117, 118]. The set of feasible solutions in this case is

$$X^{CL} = \{(x, s, z) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{Z}_+^m : s_{t-1} + x_t = d_t + s_t, t \in T; x_t \leq C_t z_t, t \in T\}.$$

We note that in many studies the special case of binary z_t variables is considered [117, 118]. The mixed n -step MIR inequalities (3.54) and (3.55) can be easily specialized to X^{CL} . Assume $\{\alpha_1, \dots, \alpha_n\}$ is the set of distinct capacity values, i.e. for any $t \in T$, $C_t = \alpha_j$ for some $j \in \{1, \dots, n\}$. So without loss of generality we assume the parameter vector is $\alpha = (\alpha_1, \dots, \alpha_n)$. Then the only difference in the above derivation is that (3.52) becomes $s_{k-1} + \sum_{t \in \{k, \dots, n_i-1\} \setminus S_i} x_t + \sum_{t \in S_i} C_t z_t \geq b_i$, and therefore in (3.54) and (3.55), we must set $y_j^i = \sum_{t \in S_i: C_t = \alpha_j} z_t$ for $i \in I, j = 1, \dots, n$. \square

Considering an $i \in I$, recall that $b_i = \sum_{t=k}^{n_i-1} d_t$, i.e. b_i is the total demand in periods k to $n_i - 1$. The n -step MIR conditions on b_i and the module capacities $\alpha_1, \dots, \alpha_n$, i.e.

$$\alpha_j \left\lceil b_i^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}, \quad j = 2, \dots, n, \quad (3.56)$$

which are required for validity of (3.54) and (3.55) have an interesting interpretation. First note that for $j = 2, \dots, n$, we have $b_i^{(j)} > 0$ and $\alpha_j > 0$, and therefore $\left\lceil b_i^{(j-1)} / \alpha_j \right\rceil \geq 1$. This along with (3.56) means the module capacities must be in non-increasing order, i.e. $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$. Now given a $j \in \{2, \dots, n\}$, consider a strategy to create the capacity required to satisfy the demand b_i that only uses modules $\alpha_1, \dots, \alpha_j$, and works as follows: We start with the largest module (i.e. α_1) and switch to opening units of the next (smaller) module only if opening another unit of

the current module makes the total opened capacity greater than b_i . We repeat this process until we reach this situation for module α_j , in which case we simply open one more unit of module α_j to make the total opened capacity greater than b_i , and stop.

It is easy to see that this strategy means setting $y_l^i = \lfloor b_i^{(l-1)}/\alpha_l \rfloor, l = 1, \dots, j-1$, $y_j^i = \lfloor b_i^{(j-1)}/\alpha_j \rfloor$, and $y_l^i = 0, l = j+1, \dots, n$. Let O_j denote the total capacity opened in this strategy. The following proposition provides an interesting interpretation for the conditions (3.56):

Proposition 33. Conditions (3.56) are equivalent to having $O_1 \geq O_2 \geq \dots \geq O_n$.

Proof. From strategy above it is easy to see that for any $j \in \{2, \dots, n\}$, we have $O_j = \sum_{l=1}^{j-1} \alpha_l \lfloor b_i^{(l-1)}/\alpha_l \rfloor + \alpha_j \lfloor b_i^{(j-1)}/\alpha_j \rfloor$. This implies $O_{j-1} \geq O_j$ is equivalent to $\alpha_j \lfloor b_i^{(j-1)}/\alpha_j \rfloor \leq \alpha_{j-1}$. Therefore, conditions (3.56) are equivalent to $O_1 \geq O_2 \geq \dots \geq O_n$. \square

Based on Proposition 33, the n -step MIR conditions (3.56) mean that the module capacities $\alpha_1, \dots, \alpha_n$ should be such that if we consider more of them in the strategy above (i.e. we increase j), the total opened capacity for covering the demand b_i using this strategy turns out to be smaller or remains the same.

Example 2. Consider the MML with two capacity modules $\alpha = (\alpha_1, \alpha_2) = (9, 4)$ and 6 time periods with demands $(d_1, d_2, d_3, d_4, d_5, d_6) = (4, 10, 17, 6, 1, 11)$. Now let $k = 2$, $l = 6$ and choose $S = \{3, 5, 6\}$ and $I = \{3, 5\}$. Therefore $S_3 = \{3\}$, $S_5 = \{3, 5\}$, $n_3 = 5$, $n_5 = 6$, $b_3 = 33$, $b_5 = 34$. The base inequalities (3.53) corresponding to time periods $i = 3$ and $i = 5$ are

$$9y_1^3 + 4y_2^3 + v_3 \geq 33,$$

$$9y_1^5 + 4y_2^5 + v_5 \geq 34,$$

where $v_3 = v_5 = s_1 + x_2 + x_4$, $y_1^3 = z_3^1$, $y_2^3 = z_3^2$, $y_1^5 = z_3^1 + z_5^1$ and $y_2^5 = z_3^2 + z_5^2$. Note

that we have $b_3^{(1)} = 6$, $b_5^{(1)} = 7$, $b_3^{(2)} = 2$, $b_5^{(2)} = 3$, and $\lceil b_3^{(1)}/\alpha_2 \rceil = \lceil b_5^{(1)}/\alpha_2 \rceil = 2$. We see that the conditions $\alpha_2 \lceil b_i^{(1)}/\alpha_2 \rceil \leq \alpha_1$, $i = 3, 5$, are satisfied. Therefore, the type I and type II mixed 2-step MIR inequalities obtained from mixing the two base inequalities are (note that $b_3^{(2)} < b_5^{(2)}$):

$$\bar{v} \geq 2(8 - y_2^3 - 2y_1^3) + (8 - y_2^5 - 2y_1^5),$$

$$\bar{v} \geq 2(8 - y_2^3 - 2y_1^3) + (8 - y_2^5 - 2y_1^5) + (8 - y_2^3 - 2y_1^3 - 1),$$

respectively, where $\bar{v} = s_1 + x_2 + x_4$. Written in terms of the original variables, these inequalities are

$$s_1 + x_2 + x_4 \geq 2(8 - z_3^2 - 2z_3^1) + (8 - z_3^2 - z_5^2 - 2z_3^1 - 2z_5^1),$$

$$s_1 + x_2 + x_4 \geq 2(8 - z_3^2 - 2z_3^1) + (8 - z_3^2 - z_5^2 - 2z_3^1 - 2z_5^1) + (8 - z_3^2 - 2z_3^1 - 1).$$

III.4.2 Multi-Module Facility Location (MMF)

We first define the multi-module facility location problem (MMF). Let $P := \{1, \dots, n_P\}$ be a set of potential facilities, $Q := \{1, \dots, n_Q\}$ be a set of clients, and $\{\alpha_1, \dots, \alpha_n\}$ be the set of capacities for n capacity modules. In MMF the goal is to decide the capacity of facilities and assign the demand of clients to facilities such that the summation of capacity setup costs and distribution costs is minimized while the demands and the capacity constraints are satisfied. The capacity of each facility is the summation of some integer multiples of $\alpha_1, \dots, \alpha_n$. Let x_{pq} be the portion of demand of client q satisfied by facility p , and u_p^j be the number of capacity modules installed in facility p . Then MMF is $\min\{\sum_{p \in P} \sum_{q \in Q} c_{pq}x_{pq} + \sum_{p \in P} f_p^j u_p^j : (x, u) \in X^{MMF}\}$, where

$$X^{MMF} = \left\{ (x, u) \in \mathbb{R}_+^{n_P n_Q} \times \mathbb{Z}_+^{n_P n} : \right.$$

$$\sum_{p \in P} x_{pq} = d_q, \quad q \in Q \quad (3.57)$$

$$\left. \sum_{q \in Q} x_{pq} \leq \sum_{j=1}^n \alpha_j u_p^j, \quad p \in P \right\}, \quad (3.58)$$

and d_q , c_{pq} , and f_p^j are the demand of client q , the distribution cost per unit between facility p and client q , and the setup cost per module of capacity α_j , $j = 1, \dots, n$, in facility p , respectively.

Let $I := \{1, 2, \dots, n_I\}$, and for $i \in I$, choose $S_i \subseteq P$ and $K_i \subseteq Q$. Let $b_i := \sum_{q \in K_i} d_q$ be the total demand of clients in K_i .

Adding the demand constraints (3.57) for $q \in K_i$, we get

$$\sum_{p \in P} w_p^i = b_i \quad (3.59)$$

where $w_p^i = \sum_{q \in K_i} x_{pq}$ is the total demand of clients in K_i satisfied by facility p . Now by (3.58), we have $w_p^i \leq \sum_{j=1}^n \alpha_j u_p^j$. Therefore for $p \in S_i$, we relax w_p^i in (3.59) to its upper bound to get

$$\sum_{p \in P \setminus S_i} w_p^i + \sum_{p \in S_i} \sum_{j=1}^n \alpha_j u_p^j \geq b_i, \quad i \in I. \quad (3.60)$$

When there is only one module size, i.e. $\alpha_j = C$, $j = 1, \dots, n$, the capacity constraints (3.58) simplify to $\sum_{q \in Q} x_{pq} \leq C y_p$, $p \in P$, where $y_p = \sum_{j=1}^n u_p^j$ (variables u_p^j are not needed anymore), and MMF reduces to CCF. We denote the feasible set of CCF by X^{CCF} . The special case of X^{CCF} where y_p , $p \in P$ are restricted to be binary was discussed in Section II.4.

Here we show that the mixed n -step MIR inequalities can be used to get valid inequalities for X^{MMF} . These inequalities generalize the inequalities presented in [70]

for X^{CCF} to the case of multiple capacities. Defining $v_i := \sum_{p \in P \setminus S_i} w_p^i$ and $y_j^i := \sum_{p \in S_i} u_p^j$, for $i \in I$, inequality (3.60) becomes

$$v_i + \sum_{j=1}^n \alpha_j y_j^i \geq b_i, \quad i \in I. \quad (3.61)$$

Notice that $v_i \in \mathbb{R}_+$, $y_j^i \in \mathbb{Z}_+$, $i \in I, j = 1, \dots, n$. Without loss of generality assume the parameter vector for mixed n -step MIR is $\alpha = (\alpha_1, \dots, \alpha_n)$ and also the indices in I are such that $b_{i-1}^{(n)} \leq b_i^{(n)}$, $i \in I$. Now if $\alpha_j \left\lceil b_i^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}$, $j = 2, \dots, n, i \in I$, by letting $\bar{v} = \sum_{(p,q) \in T} x_{pq}$, where $T = \{(p, q) : p \in P \setminus S_i, q \in K_i \text{ for some } i \in I\}$ (note that $\bar{v} \geq v_i$ for all $i \in I$), based on Corollary 22, the mixed n -step MIR inequalities

$$\sum_{(p,q) \in T} x_{pq} \geq \sum_{i=1}^{n_I} \left(b_i^{(n)} - b_{i-1}^{(n)} \right) \phi^i(y^i), \quad (3.62)$$

$$\sum_{(p,q) \in T} x_{pq} \geq \sum_{i=1}^{n_I} \left(b_i^{(n)} - b_{i-1}^{(n)} \right) \phi^i(y^i) + \left(\alpha_n - b_{n_I}^{(n)} \right) \left(\phi_n^1(y^1) - 1 \right) \quad (3.63)$$

are valid for X^{MMF} , where $y_j^i = \sum_{p \in S_i} u_p^j$. The inequalities for X^{CCF} presented in [70] are the special case of (3.62) for $n = 1$ (the constant capacity case).

Remark 3. A special case of MMF is when each facility p can have only modules of a specific capacity C_p but the capacity of modules in different facilities are not necessarily the same. This is the well-known capacitated facility location problem (CF) [1, 2, 117]. The set of feasible solutions in this case is

$$X^{CF} = \left\{ (x, u) \in \mathbb{R}_+^{n_P n_Q} \times \mathbb{Z}_+^{n_P} : \sum_{p \in P} x_{pq} = d_q, q \in Q; \sum_{q \in Q} x_{pq} \leq C_p u_p, p \in P \right\}.$$

We note that in many studies the special case of binary u_p variables is considered [1, 2, 117].

The mixed n -step MIR inequalities (3.62) and (3.63) can be easily specialized to X^{CF} very similar to the way (3.54) and (3.55) were specialized to X^{CL} in Remark 2 with $y_j^i = \sum_{p \in S_i: C_p = \alpha_j} u_p$ for $i \in I, j = 1, \dots, n$. \square

Considering an $i \in I$, the n -step MIR conditions on the demand b_i and the module capacities $\alpha_1, \dots, \alpha_n$, i.e. $\alpha_j \left\lceil b_i^{(j-1)} / \alpha_j \right\rceil \leq \alpha_{j-1}, j = 2, \dots, n$, which are required for validity of (3.62) and (3.63) have an interpretation similar to the one described in Section III.4.1.

III.5 Computational Results

In this section, we present our preliminary computational results on using the mixed n -step MIR inequalities for general MIP in solving small MIPLIB instances as well as using the mixed n -step MIR inequalities (3.54) in solving multi-module lot-sizing (MML) instances.

III.5.1 MIPLIB Instances

In the first part of our computational study, we compared the performance of three family of cuts, namely MIR (i.e. 1-step MIR), 2-row mixed 1-step MIR, and 2-row mixed 2-step MIR, on small MIPLIB instances. It is known that the separation problem for MIR cuts is strongly NP-complete [34], so naturally, one does not expect existence of an efficient exact separation algorithm for the MIR cuts. The complexity and existence of an efficient exact separation for the n -step MIR cuts for $n \geq 2$, and the mixed n -step MIR cuts for $n \geq 1$, are open problems. These problems have not been addressed even for the 2-step MIR [46] and the mixed 1-step MIR [70], which were introduced before n -step MIR [78] (we note that Dash and Günlük [47] formulated the separation problem for the mixed 1-step MIR cuts as mixed integer programs). Given the more complicated structure of n -step MIR and mixed n -step MIR cuts, the exact separation problems for these cuts and determining their complexity do not seem to be easy.

As a result, in our study we used a heuristic separation algorithm based on the ideas

of the heuristic proposed by Marchand and Wolsey [90] for 1-step MIR cuts. To our knowledge, this separation heuristic (or its variants) is the only existing heuristic which works well for application of general purpose MIR-based cuts on instances such as those in MIPLIB, which are generally quite sparse and have bounds on a large number of integer variables. The aggregation and bound substitution elements of this heuristic provide suitable base inequalities to apply n -step MIR functions. The details of our separation heuristic are as follows:

We used the aggregation and bound substitution heuristics of [90] to generate the base inequalities for which the cuts are developed. Given an instance and the optimal solution of its LP relaxation, we converted the constraints of the problem to equality constraints by adding necessary slack variables and used the aggregation heuristic of [90] to aggregate the constraints of the problem according to the procedure presented in [90] (the MAXAGGR parameter of [90] was set to 6). We then applied criterion (a) of the bound substitution heuristic in [90] (which uses the optimal LP relaxation solution) to generate base constraints of the form of the defining constraints of Y_m .

For each instance we performed three experiments. In each experiment, the cuts were generated only at the root node and from the base constraints developed as explained above. In the first experiment, denoted by 1MIR, we added only 1-step MIR cuts to the problem. For each base constraint, we generated the 1-step MIR cuts (see Section II.2) by setting the parameter α_1 equal to each one of the positive coefficients of integer variables in the base constraint and added those cuts that were violated by the optimal LP relaxation solution to the problem.

In the second experiment, denoted by 1MIR1MIX, we added mixed 1-step MIR cuts *in addition to* the 1-step MIR cuts that were added in experiment 1MIR. More specifically, after adding the cuts of 1MIR, we re-optimized the LP relaxation and used the new LP relaxation solution in separation with mixed 1-step MIR cuts. The

mixed 1-step MIR cuts were generated from the same base constraints resulted from aggregation and bound substitution procedure above. We only considered 2-row mixing ($k = 2$). All pairs of the base constraints were considered for mixing. For each pair, we generated a set of mixed 1-step MIR cuts according to Theorem 32 (we used $\bar{s} = s_1 + s_2$) by setting the value of the parameter α_1 equal to each one of the positive coefficients of integer variables in the two base constraints. Out of all the cuts generated by these choices of α_1 , we added to the problem those that were violated by the optimal LP relaxation solution.

The third experiment, denoted by (1MIR2MIX), is similar to 1MIR1MIX, however we added mixed 2-step MIR cuts (Section III.3) instead of mixed 1-step MIR cuts. The details are the same as 1MIR1MIX. The only difference is in choosing parameters α_1 and α_2 . For each pair of the base constraints, we constructed a list consisting of all positive coefficients of integer variables in the two base inequalities and then considered all pairs of parameters from this list that satisfy the 2-step MIR condition, i.e. conditions (3.40) for $n = k = 2$. Out of all the cuts generated by these choices of α_1 and α_2 , we added to the problem those that were violated by the LP relaxation solution.

We note that in the experiments above, our method of choosing values for the parameters α_1 and α_2 (choosing from the coefficients of base constraints) was motivated by the facet-defining conditions for the n -step MIR inequalities presented in [18].

We limited our experiments to small instances in MIPLIB libraries. More specifically, we selected all instances from MIPLIB 3.0, 2003, and 2010 which have less than 40 rows and less than 1000 columns. Out of these instances, we ignored one infeasible instance (`p2m2p1m1p0n100` from MIPLIB 2010) as well as the following instances: `enigma` from MIPLIB 3.0 because it has an integrality gap of zero as well as `markshare1` and `markshare2` from MIPLIB 2003, and `markshare_5_0` from MIPLIB 2010, because

their solution time using CPLEX 11.0 even with no cuts was prohibitively long. This left us with 8 instances which are from MIPLIB 3.0 and 2003.

In all three experiments, we solved the LP relaxation after adding the cuts and found its optimal solution. We then dropped the cuts that were inactive at this optimal solution and solved the MIP with active cuts. We also solved the LP relaxation and MIP with no cuts for all instances, denoted by NOCUTS. We used CPLEX 11.0 with its default options. The program was coded in Microsoft Visual C++ and run on a PC with Intel Quad Core 2.4GHz processor with 4MB of RAM. The results are presented in Table I. The *cuts* row shows the number of 1-step MIR cuts in 1MIR, number of mixed 1-step MIR cuts (in addition to 1-step MIR cuts) in 1MIR1MIX, and number of mixed 2-step MIR (in addition to 1-step MIR cuts) in 1MIR2MIX. The *nodes* and *time* rows show the number of branch-and-bound nodes and time (in seconds) to solve the MIP to optimality. The *gapclosed* row shows the percentage of the integrality gap closed by the cuts in each experiment, i.e. $gapclosed = 100(z_{cut} - z_{lp}) / (z_{mip} - z_{lp})$, where z_{lp} , z_{cut} , and z_{mip} are the optimal objective values of the LP relaxation with no cuts, LP relaxation with the cuts, and MIP, respectively.

Comparing the percentage of integrality gap that is closed among the three experiments, we see that in all instances except **flugpl**, for which our separation did not results in any mixed 1-step or 2-step MIR cut, adding mixed 1-step MIR cuts over 1-step MIR cuts has improved the closed gap. The maximum improvement is $36.10\% - 24.44\% = 11.66\%$ (for **mod008**). More interestingly, in these instances adding mixed 2-step MIR cuts over 1-step MIR cuts has improved the closed gap more than adding mixed 1-step MIR cuts over 1-step MIR cuts. For 1MIR2MIX, the maximum improvement over 1MIR is $44.99\% - 24.44\% = 20.55\%$ (for **mod008**).

Table I. Results of computational experiments on small MIPLIB instances

	Instance	flugpl	gt2	lseu	mas74	mas76	mod008	p0033	rgn
NOCUTS	<i>zlp</i>	1167190	13460.2	834.68	10482.8	38893.9	290.93	2520.57	48.8
	<i>zmip</i>	1201500	21166	1120	11801.2	40005.1	307	3089	82.2
	<i>time</i>	0.0	0.0	0.1	280	50	0.2	0.0	0.1
	<i>nodes</i>	94	1	101	2672210	403345	577	6	523
1MIR	<i>cuts</i>	1	11	19	74	59	8	11	18
	<i>zcut</i>	1167880	20539.7	980.43	10547.5	38956.9	294.86	2829.69	56.52
	<i>time</i>	0.0	0.0	0.1	272	60	0.1	0.0	0.2
	<i>nodes</i>	67	1	155	2714275	408774	83	1	503
	<i>gapclosed</i>	2.01	91.87	51.08	4.91	5.67	24.44	54.38	23.10
1MIR1MIX	<i>cuts</i>	0	22	9	15	13	9	6	4
	<i>zcut</i>	1167880	20592.9	993.59	10554	38971.3	296.73	2834.39	56.93
	<i>time</i>	0.0	0.0	0.1	313	41	0.1	0.0	0.2
	<i>nodes</i>	67	1	136	2879937	239351	148	1	549
	<i>gapclosed</i>	2.01	92.56	55.70	5.40	6.97	36.10	55.21	24.34
1MIR2MIX	<i>cuts</i>	0	106	47	131	114	143	18	60
	<i>zcut</i>	1167880	20725.5	1000.51	10577.1	39012.8	298.16	2849.8	60.98
	<i>time</i>	0.0	0.0	0.1	366	73	0.3	0.0	0.2
	<i>nodes</i>	67	1	123	2509761	223216	71	1	606
	<i>gapclosed</i>	2.01	94.28	58.12	7.15	10.70	44.99	57.92	36.45

Note that going from 1MIR to 1MIR1MIX to 1MIR2MIX, while the closed gap has improved, in most cases the number of nodes and solution time have either significantly decreased or remained almost the same. These results are quite promising in light of the fact that MIPLIB instances are notorious with respect to gap improvement beyond what is achieved by 1-step MIR [58].

III.5.2 Multi-module Lot-sizing Instances

In the second part of our computational study, we studied the performance of the mixed 2-step MIR cuts (3.54) in solving randomly generated MML instances with two capacity modules. Here we also used a heuristic separation algorithm. Our separation is designed based on the method presented in Section III.4.1 to generate inequality (3.54). Using the notation of Section III.4.1, given an instance and the optimal solution of its LP relaxation, denoted by $(\bar{x}, \bar{s}, \bar{z})$, our heuristic is as follows: We considered all possible choices $k, l \in \{1, \dots, T\}$ such that $k < l$. For each choice of k and l , we considered three choices for S : $S = \{k, \dots, l\}$, $S = \{t \in \{k, \dots, l\} : \bar{z}_t^1 > 0 \text{ or } \bar{z}_t^2 > 0\}$,

and $S = \{t \in \{k, \dots, l\} : \bar{z}_t^1 \notin \mathbb{Z} \text{ or } \bar{z}_t^2 \notin \mathbb{Z}\}$. Similar to the previous section, we only considered 2-row mixing (i.e. $|I| = 2$). Therefore our choices for I included all possible two-element subsets of S . For each I , we generated inequality (3.54) if $\alpha_2 \left\lceil b_i^{(1)} / \alpha_2 \right\rceil < \alpha_1$ for $i \in I$ and added it as a cut if it was violated by the optimal LP relaxation solution. As before, all the cuts were added to the root node.

We created random MML instances with two capacity modules ($n = 2$) for this experiment. All our instances had 60 time periods, i.e. $T = \{1, \dots, 60\}$. The holding cost in all periods was 10, i.e. $h_t = 10, t \in T$. Demand d_t and production cost p_t in each period were integers drawn from *uniform*[10, 190] and *uniform*[81, 119], respectively. In [21] it was observed that the difficulty of capacitated lot-sizing (CL) instances is a function of tightness of the capacities with respect to the demand and the ratio of the setup cost to holding cost. Therefore, we used two sets of capacity modules: $\alpha = (\alpha_1, \alpha_2) = (180, 80)$ and $\alpha = (\alpha_1, \alpha_2) = (270, 130)$, the former resulting in harder instances than the latter. We also used two sets of setup costs for these modules: $(f_t^1, f_t^2) = (1000, 600), t \in T$, and $(f_t^1, f_t^2) = (5000, 2600), t \in T$, the former resulting in easier instances than the latter. We generated 5 instances for each combination of α and (f_t^1, f_t^2) , i.e. a total of 20 instances. We note that some of the instance generation and separation ideas we used here are inspired by the ideas used in [21] for CL problems.

For each instance, we solved the LP relaxation and MIP without adding any cuts (denoted by NOCUTS). We also solved the LP relaxation after adding the cuts, found its optimal solution, dropped the cuts that were inactive at this optimal solution, and solved the MIP with active cuts (denoted by 2MIX). The software and hardware platforms we used was the same as those used for MIPLIB instances. The results are presented in Table II. The definitions of column labels are the same as the definitions of row labels for Table I described in Section III.5.1.

Table II. Results of computational experiments on MML instances

Instance		NOCUTS					2MIX			
(α_1, α_2)	(f_t^1, f_t^2)	<i>zlp</i>	<i>zmip</i>	<i>time</i>	<i>nodes</i>	<i>cuts</i>	<i>zcut</i>	<i>time</i>	<i>nodes</i>	<i>gapclosed</i>
(180,80)	(1000,600)	559248	567703	0.3	517	729	566565	0.4	73	86.54
		646576	654258	0.2	506	509	653332	0.2	17	87.95
		615880	623663	0.1	261	443	622775	0.1	1	88.59
		612767	620872	0.0	58	589	620185	0.2	2	91.52
		571612	580115	0.2	470	607	579458	0.1	1	92.27
	(5000,2600)	761700	785624	109.7	508198	572	782166	5.0	2534	85.55
		812633	835040	53.1	228982	741	831892	7.7	1942	85.95
		831488	852734	61.2	240425	567	849985	4.8	2603	87.06
		812841	832604	30.3	145749	520	830666	0.9	399	90.19
		761053	782019	39.8	164846	570	780009	1.2	564	90.41
	(1000,600)	730889	741886	0.0	43	488	740768	0.2	22	89.83
		590107	598604	0.0	29	664	597766	0.3	9	90.14
		616219	627391	0.3	412	578	626296	0.2	1	90.20
		619897	630661	0.0	18	721	629622	0.3	22	90.35
		541672	550644	0.0	157	458	549868	0.1	1	91.35
	(5000,2600)	604703	629971	19.2	86812	742	626920	4.9	3288	87.93
		749124	774130	2.0	6809	517	771468	0.9	453	89.35
		703081	726339	0.5	1161	652	724118	0.6	123	90.45
		660877	684319	0.6	1439	651	682235	0.6	183	91.11
		669220	691974	0.6	973	612	690164	0.5	43	92.05

Table II shows that the mixed 2-step MIR cuts are very effective in solving the MML problems. The percentage of integrality gap closed by these cuts is between 85.55% and 92.27% (the average is 89.44%). We also observe that adding the cuts has reduced the number of nodes in almost all instances by several orders of magnitude, especially in harder instances (which have larger number of nodes and solution times). In harder instances, the solution time has also substantially reduced.

III.6 Concluding Remarks

We showed that mixing can be generalized to n -step MIR resulting in the mixed n -step MIR inequalities for a generalization of the mixing set called the n -mixing set. The parameters $\alpha_1, \dots, \alpha_n$ must satisfy the same conditions required for the validity of n -step MIR inequalities. As a special case these conditions are automatically satisfied if the parameters $\alpha_1, \dots, \alpha_n$ are divisible. Moreover, the type I and type II mixed n -step MIR inequalities are strong in the sense that they define facets for the n -mixing set.

We also showed that mixed n -step MIR can be used to generate cuts based on multiple constraints for general MIPs as well as multi-module lot-sizing and facility location problems. The mixed n -step MIR encompasses, as the special case corresponding to $n = 1$, the inequalities that were previously generated based on mixing of MIR inequalities for the mixing set [70] as well as lot-sizing and facility location problems with a constant capacity [1, 2, 104]. Our preliminary computational results on applying mixed n -step MIR inequalities in solving multi-module lot-sizing instances and small MIPLIB instances justify their effectiveness.

CHAPTER IV

 n -STEP CONIC MIR INEQUALITIES

In this chapter, we introduce a new class of valid inequalities for general second-order conic MIPs and linear MIPs and establish several theoretical properties for these valid inequalities. More specifically, we introduce the *n -step conic MIR inequalities* for SOCMIPs. The simple conic MIR inequalities of [19] and the n -step MIR inequalities of [78] are special cases of the n -step conic MIR inequalities. For any positive integer n , given the positive parameter vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and any $\beta \in \mathbb{R}$, define the recursive remainders $\beta^{(i)} := \beta^{(i-1)} - \alpha_i \lfloor \beta^{(i-1)} / \alpha_i \rfloor$, $i = 1, \dots, n$, where $\beta^{(0)} = \beta$. First, in Sect. IV.2, we will derive and show the validity of the n -step conic MIR inequality for a generalization of \overline{Q} with multiple integer variables, i.e. the PSOC set

$$Q^n = \left\{ (y, w^+, w^-, t) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+^3 : \left| \sum_{i=1}^n \alpha_i y_i + w^+ - w^- - \beta \right| \leq t \right\},$$

where the conditions

$$\alpha_i \lceil \beta^{(i-1)} / \alpha_i \rceil \leq \alpha_{i-1} \quad \text{for } i = 2, \dots, n \quad (4.1)$$

hold. We also show that the n -step conic MIR inequality defines a facet for $\text{conv}(Q^n)$. The conic MIR inequality of [19] for \overline{Q} is simply the special case of $n = 1$ and $\alpha_1 = 1$. In addition, we show that all the 1-step to $(n-1)$ -step conic MIR inequalities are also facet-defining for $\text{conv}(Q^n)$ if an additional simple condition is satisfied.

The n -step conic MIR inequality is a linear inequality. In Sect. IV.3, we will prove that the polyhedral second-order conic form of this inequality is valid for the set Q if and only if conditions (4.1) hold at equality. Such an inequality can be used in developing nonlinear inequalities for conic mixed integer sets of appropriate form.

Next in Sect. IV.4, we use the n -step conic MIR facet for Q^n to develop the n -step conic MIR inequality for the general PSOC set S . We do this by developing a superadditive function, which we refer to as the *n -step conic MIR function*. The right-hand side b and a choice of n parameters $\alpha_1, \dots, \alpha_n$, which satisfy conditions (4.1), completely define an instance of this function. The n -step conic MIR inequality for S is generated by applying the n -step conic MIR function on a_j 's and b . The conic MIR function of [19] is the special case of $n = 1$. Moreover, we will prove that n -step conic MIR inequalities define facets for $\text{conv}(S)$ under simple conditions.

We further prove in Section IV.5 that the n -step conic MIR inequality for the set S strictly dominates the n -step MIR inequalities that are written for the two linear constraints that define S . As a result, n -step conic MIR is not simply obtainable using n -step MIR. We also show that the n -step MIR inequality of Kianfar and Fathi [78] for the set Y is a special case of n -step conic MIR inequality. We conclude in Section IV.6 with a few remarks.

We start by presenting some preliminary results, which will be used in our developments throughout the chapter in Section IV.1.

IV.1 Preliminaries

In this section, we prove some preliminary lemmas that will be helpful in developing our main results in the next sections. In the rest of this chapter, we make the general assumption that $b^{(i-1)}/\alpha_i \notin \mathbb{Z}$, $i = 1, \dots, n$ (or in other words $b^{(i)} \neq 0$, $i = 0, 1, \dots, n$)

because otherwise the n -step conic MIR inequality reduces to a trivial inequality.

We first present a lemma which allows us to easily handle variables w^+ and w^- in developing valid inequalities for the set Q^n and S . Let Q_0^n be the set obtained by dropping w^+ and w^- from Q^n , i.e.

$$Q_0^n := \left\{ (y, t) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+ : \left| \sum_{i=1}^n \alpha_i y_i - \beta \right| \leq t \right\}.$$

Lemma 34. *Inequality $\sum_{i=1}^n \pi_i y_i + \pi_0 \leq t$ is valid for Q_0^n if and only if inequality $\sum_{i=1}^n \pi_i y_i + \pi_0 \leq t + w^+ + w^-$ is valid for Q^n .*

Proof. First assume $\sum_{i=1}^n \pi_i y_i + \pi_0 \leq t$ is valid for Q_0^n . Consider any $(y, w^+, w^-, t) \in Q^n$. We have

$$\begin{aligned} t &\geq \left| \sum_{i=1}^n \alpha_i y_i + w^+ - w^- - \beta \right| \\ &\geq \left| \sum_{i=1}^n \alpha_i y_i - \beta \right| - |w^+ - w^-| \\ &\geq \left| \sum_{i=1}^n \alpha_i y_i - \beta \right| - w^+ - w^-. \end{aligned}$$

The last inequality is true because w^+ and w^- are nonnegative. Therefore, we have $t + w^+ + w^- \geq \left| \sum_{i=1}^n \alpha_i y_i - \beta \right|$, which means $(y, t + w^+ + w^-) \in Q_0^n$. Therefore based on the assumption, we get $\sum_{i=1}^n \pi_i y_i + \pi_0 \leq t + w^+ + w^-$. The other direction is trivial because $(y, t) \in Q_0^n$ means $(y, 0, 0, t) \in Q^n$ implying $\sum_{i=1}^n \pi_i y_i + \pi_0 \leq t$. \square

Using Lemma 34, we can develop valid inequalities for the simpler set Q_0^n and easily extend them to valid inequalities for the set Q^n . Next, we prove two other lemmas, which will be helpful in our later developments in this chapter.

Lemma 35. *The inequality $\sum_{i=1}^n \pi_i y_i + \pi_0 \leq t$ is valid for Q_0^n if and only if $\sum_{i=1}^n \pi_i y_i + \pi_0 \leq \left| \sum_{i=1}^n \alpha_i y_i - \beta \right|$ is valid for Q_0^n .*

Proof. Assume $\sum_{i=1}^n \pi_i y_i + \pi_0 \leq t$ is valid for Q_0^n . Observe that if $\sum_{i=1}^n \pi_i \bar{y}_i + \pi_0 > \left| \sum_{i=1}^n \alpha_i \bar{y}_i - \beta \right|$ for some $(\bar{y}, \bar{t}) \in Q_0^n$, then a point (\bar{y}, \hat{t}) exists where $\left| \sum_{i=1}^n \alpha_i \bar{y}_i - \beta \right| \leq \hat{t} < \sum_{i=1}^n \pi_i \bar{y}_i + \pi_0$. This point belongs to Q_0^n but violates $\sum_{i=1}^n \pi_i y_i + \pi_0 \leq t$, which is a contradiction. The other direction is trivial. \square

Lemma 36. For $n, l \in \mathbb{N}$, where $l \leq n$, the following identity is true:

$$\prod_{k=l}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil = \sum_{i=l}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor + 1. \quad (4.2)$$

Proof. This is true because

$$\begin{aligned} \prod_{k=l}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil &= \prod_{k=l+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{\beta}{\alpha_1} \right\rfloor + \prod_{k=l+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \\ &= \prod_{k=l+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{\beta}{\alpha_1} \right\rfloor + \prod_{k=l+2}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{\beta^{(1)}}{\alpha_2} \right\rfloor + \prod_{k=l+2}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \\ &= \dots = \sum_{i=l}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor + 1. \end{aligned}$$

\square

A helpful result is the following lemma presented and proved in [17]:

Lemma 37. [17] Let $K_{\geq} = \{(x, s) \in \mathbb{Z}^n \times \mathbb{R}_+ : ax + s \geq b\}$ and $K_{\leq} = \{(y, t) \in \mathbb{Z}^n \times \mathbb{R}_+ : ay - t \leq b\}$. The inequality $\pi x + s \geq \pi_0$ is valid for K_{\geq} if and only if the inequality $(a - \pi)y - t \leq b - \pi_0$ is valid for K_{\leq} . Moreover, $\pi x + s \geq \pi_0$ is facet-defining for $\text{conv}(K_{\geq})$ if and only if $(a - \pi)y - t \leq b - \pi_0$ is facet-defining for $\text{conv}(K_{\leq})$.

As mentioned in Section II.5, Atamtürk and Narayanan [19] presented the so-called simple conic MIR cut for the set \bar{Q} . In this chapter, we refer to this inequality as the 1-step conic MIR inequality because as we will see it is the special case of the n -step conic MIR for $n = 1$. Also in this chapter, we consider this inequality for the set $Q^1 = \{(y_1, w^+, w^-, t) \in \mathbb{Z} \times \mathbb{R}_+^3 : |\alpha_1 y_1 + w^+ - w^- - \beta| \leq t\}$, which is equivalent to the

set \overline{Q} but has a form more suitable for our generalization to n -step conic MIR.

Here we introduce an alternative proof for the validity of the 1-step conic MIR inequality, which is more straightforward than the proof in [19], and more importantly, inspires the proof of our generalization to n -step conic MIR presented in Sect. IV.2.

Lemma 38. *The 1-step conic MIR inequality*

$$(\alpha_1 - 2\beta^{(1)})(y_1 - \lfloor \beta/\alpha_1 \rfloor) + \beta^{(1)} \leq t + w^+ + w^- \quad (4.3)$$

is valid for Q^1 and defines a facet for $\text{conv}(Q^1)$.

Proof. Consider the set $Q_0^1 = \{(y_1, t) \in \mathbb{Z} \times \mathbb{R}_+ : |\alpha_1 y_1 - \beta| \leq t\}$. Let $(y_1, t) \in Q_0^1$. We consider two possible cases: First assume $\alpha_1 y_1 - \beta > 0$. So since $y_1 \in \mathbb{Z}$, we have $y_1 - \lfloor \beta/\alpha_1 \rfloor \geq 1$ (recall the general assumption that $b/\alpha_1 \notin \mathbb{Z}$). Multiplying this inequality by $b^{(1)}$ (which is nonnegative), we get

$$\beta^{(1)}(y_1 - \lfloor \beta/\alpha_1 \rfloor) - \beta^{(1)} \geq 0. \quad (4.4)$$

On the other hand, in this case by the defining inequality of Q_0^1 we have $\alpha_1 y_1 - \beta \leq t$, which can be written as

$$\alpha_1(y_1 - \lfloor \beta/\alpha_1 \rfloor) - \beta^{(1)} \leq t. \quad (4.5)$$

Multiplying inequality (4.4) by -2 and adding it to inequality (4.5) yields

$$(\alpha_1 - 2\beta^{(1)})(y_1 - \lfloor \beta/\alpha_1 \rfloor) + \beta^{(1)} \leq t. \quad (4.6)$$

Now, consider the case where $\alpha_1 y_1 - \beta < 0$. Since $y_1 \in \mathbb{Z}$, we have $y_1 - \lfloor \beta/\alpha_1 \rfloor \leq 0$. Multiplying this inequality by $\alpha_1 - \beta^{(1)}$ (which is positive), we get

$$(\alpha_1 - \beta^{(1)})(y_1 - \lfloor \beta/\alpha_1 \rfloor) \leq 0. \quad (4.7)$$

This result is partly due to Sina Masihabadi.

In this case, by the defining inequality of Q_0^1 , we have $-\alpha_1 y_1 + \beta \leq t$, which can be written as

$$-\alpha_1 (y_1 - \lfloor \beta/\alpha_1 \rfloor) + \beta^{(1)} \leq t. \quad (4.8)$$

Multiplying inequality (4.7) by 2 and adding it to inequality (4.8) gives (4.6). Hence, inequality (4.6) is valid for Q_0^1 , which based on Lemma 34, implies validity of (4.3) for Q^1 .

The proof that (4.3) is also facet-defining for $\text{conv}(Q^1)$ is similar to the proof in [19] and due to the fact the four affinely independent points $p_1^1 = (\lceil \beta/\alpha_1 \rceil, 0, 0, \alpha_1 - \beta^{(1)})$, $q_1^1 = (\lfloor \beta/\alpha_1 \rfloor, 0, 0, \beta^{(1)})$, $r^1 = (\lfloor \beta/\alpha_1 \rfloor, \beta^{(1)}, 0, 0)$, and $s^1 = (\lfloor \beta/\alpha_1 \rfloor, 0, \alpha_1 - \beta^{(1)}, 0)$ belong to Q^1 and satisfy (4.3) at equality (the points are in the form (y_1, w^+, w^-, t)). \square

IV.2 n -step Conic MIR Facet for Q^n

In this section, we introduce the n -step conic MIR inequalities for the set Q^n . We show that for any $n \in \mathbb{N}$, this inequality is valid for Q^n and defines a facet for its convex hull if conditions (4.1) are satisfied. This presents a generalization of the result in Lemma 38. The conic MIR inequality of [19] for the set \bar{Q} or Q^1 is the special case of $n = 1$ (hence called the 1-step conic MIR in this chapter). Moreover, we show that the n_1 -step conic MIR inequality, where $n_1 < n$, is also valid and facet-defining for $\text{conv}(Q^n)$.

Theorem 39. *If conditions (4.1) hold, the n -step conic MIR inequality*

$$\sum_{i=1}^n \left(\alpha_i - 2\beta^{(n)} \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) + \beta^{(n)} \leq t + w^+ + w^- \quad (4.9)$$

is valid for Q^n .

This result is partly due to Sina Masihabadi.

Proof. Consider the set Q_0^n and any point $q = (y, t) \in Q_0^n$. We consider two cases for this point: First assume $\sum_{i=1}^n \alpha_i y_i \geq \beta$. This is the defining inequality of P^n with $v = 0$. Since conditions (4.1) are satisfied, q satisfies inequality (2.6) with $v = 0$, i.e.

$$\beta^{(n)} \sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil y_i \geq \beta^{(n)} \prod_{k=1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil. \quad (4.10)$$

Replacing for the expression $\prod_{k=1}^n \lceil \beta^{(k-1)} / \alpha_k \rceil$ in the right-hand side of (4.10) using Lemma 36, inequality (4.10) can be written as

$$\beta^{(n)} \sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) - \beta^{(n)} \geq 0. \quad (4.11)$$

On the other hand, based on the defining inequality of Q_0^n , in this case $\sum_{i=1}^n \alpha_i y_i - \beta \leq t$, which can be written as

$$\sum_{i=1}^n \alpha_i (y_i - \lfloor \beta^{(i-1)} / \alpha_i \rfloor) - \beta^{(n)} \leq t. \quad (4.12)$$

Multiplying (4.11) by -2 and adding it to (4.12) yields

$$\sum_{i=1}^n \left(\alpha_i - 2\beta^{(n)} \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) + \beta^{(n)} \leq t. \quad (4.13)$$

Now, consider the second case, i.e. when $\sum_{i=1}^n \alpha_i y_i \leq \beta$. This is the defining inequality of P^n , where the direction of the inequality is reversed and $v = 0$. Since conditions (4.1) hold, based on Lemma 37 and inequality (2.6), q must satisfy

$$\sum_{i=1}^n \left(\alpha_i - \beta^{(n)} \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) y_i \leq \beta - \beta^{(n)} \prod_{k=1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil. \quad (4.14)$$

Replacing for the expression $\prod_{k=1}^n \lceil \beta^{(k-1)} / \alpha_k \rceil$ in the right-hand side of (4.14) from Lemma 36 and using the identity $\beta = \sum_{i=1}^n \alpha_i \lfloor \beta^{(i-1)} / \alpha_i \rfloor + \beta^{(n)}$, inequality (4.14) can

be written as

$$\sum_{i=1}^n \left(\alpha_i - \beta^{(n)} \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) \leq 0. \quad (4.15)$$

On the other hand, based on the defining inequality of Q_0^n , in this case $-\sum_{i=1}^n \alpha_i y_i + \beta \leq t$, which can be written as

$$-\sum_{i=1}^n \alpha_i (y_i - \lfloor \beta^{(i-1)} / \alpha_i \rfloor) + \beta^{(n)} \leq t. \quad (4.16)$$

Multiplying (4.15) by 2 and adding it to (4.16) gives (4.13). Hence, (4.13) is satisfied by q in both cases. Therefore inequality (4.13) is valid for Q_0^n . This along with Lemma 34 implies the n -step conic MIR inequality (4.9) is valid for Q^n . \square

Note that for $n = 1$, Theorem 39 reduces to Lemma 38. Next we show that lower-step conic MIR inequalities are also valid for Q^n .

Corollary 40. *Let $n_1 \leq n$. If $\alpha_i \lceil \beta^{(i-1)} / \alpha_i \rceil \leq \alpha_{i-1}$ for $i = 2, \dots, n_1$. Then the following inequality is valid for Q^n :*

$$\begin{aligned} \sum_{i=1}^{n_1} \left(\alpha_i - 2\beta^{(n_1)} \prod_{k=i+1}^{n_1} \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) + \beta^{(n_1)} \\ \leq t + \sum_{i=n_1+1}^n \alpha_i y_i + w^+ + w^-. \end{aligned} \quad (4.17)$$

Proof. For $n_1 = n$, the corollary is the same as Theorem 39 (inequality (4.17) reduces to inequality (4.9)). So consider $n_1 < n$. By Theorem 39, the n_1 -step conic MIR inequality (i.e. inequality (4.9) for $n = n_1$) is valid for Q^{n_1} . Notice that for any point in Q^n , $\sum_{i=n_1+1}^n \alpha_i y_i + w^+ \in \mathbb{R}_+$. Therefore $\sum_{i=n_1+1}^n \alpha_i y_i + w^+$ in Q^n can be treated as w^+ in Q^{n_1} . Thus the n_1 -step conic MIR inequality for Q^{n_1} , where w^+ is replaced with $\sum_{i=n_1+1}^n \alpha_i y_i + w^+$, i.e. inequality (4.17), is valid for Q^n . \square

In the following, we show that the n -step conic MIR inequality (4.9) as well as inequalities (4.17) are also facet-defining for $\text{conv}(Q^n)$. We do so by identifying $n + 3$ affinely independent feasible points that lie on the face defined by these valid inequalities. To this end, we first define a collection of points and prove some properties for them.

Definition 41. For $n \in \mathbb{N}$ and $k = 1, \dots, n$, we define the points $p_k^n, q_k^n, r^n, s^n = (y_1, \dots, y_n, w^+, w^-, t) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+^3$ as follows:

- For p_k^n :

$$y_i = \begin{cases} \lfloor \beta^{(i-1)} / \alpha_i \rfloor & \text{for } i = 1, \dots, k-1 \\ \lceil \beta^{(i-1)} / \alpha_i \rceil & \text{for } i = k \\ 0 & \text{for } i = k+1, \dots, n \end{cases}$$

$$w^+ = w^- = 0, \text{ and } t = \alpha_k - \beta^{(k)}.$$

- For q_k^n :

$$y_i = \begin{cases} \lfloor \beta^{(i-1)} / \alpha_i \rfloor & \text{for } i = 1, \dots, k \\ 0 & \text{for } i = k+1, \dots, n \end{cases}$$

$$w^+ = w^- = 0, \text{ and } t = \beta^{(k)}.$$

- For r^n : $y_i = \lfloor \beta^{(i-1)} / \alpha_i \rfloor$, $i = 1, \dots, n$, $w^+ = \beta^{(n)}$ and $w^- = t = 0$.

- For s^n : $y_1 = \lceil \beta / \alpha_1 \rceil$, $y_i = 0$, $i = 2, \dots, n$, $w^+ = t = 0$, and $w^- = \alpha_1 - \beta^{(1)}$.

Lemma 42. For $n \in \mathbb{N}$, the points p_k^n, q_k^n , $k = 1, \dots, n$, r^n , and s^n are in Q^n .

Proof. It is clear that all the points are in $\mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+^3$. Now substituting the coordinates of p_k^n into the defining inequality of Q^n gives

$$\left| \sum_{i=1}^{k-1} \alpha_i \lfloor \beta^{(i-1)} / \alpha_i \rfloor + \alpha_k \lceil \beta^{(k-1)} / \alpha_k \rceil - \beta \right| \leq \alpha_k - \beta^{(k)},$$

or $|\alpha_k - \beta^{(k)}| \leq \alpha_k - \beta^{(k)}$, which is true since $\alpha_k - \beta^{(k)} \geq 0$. Substituting the coordinates of q_k^n into the defining inequality of Q^n gives $|\sum_{i=1}^k \alpha_i \lfloor \beta^{(i-1)} / \alpha_i \rfloor - \beta| \leq \beta^{(k)}$, or $|\beta^{(k)}| \leq \beta^{(k)}$, which is true since $\beta^{(k)} \geq 0$ for $k = 1, \dots, n$. It is very easy to verify that substituting the coordinates of r^n and s^n into the defining inequality of Q^n results in trivial inequalities. \square

Lemma 43. *Let $n, n_1 \in \mathbb{N}$, where $n_1 \leq n$. The points $p_1^n, \dots, p_{n_1}^n, q_{n_1}^n, \dots, q_n^n, r^n$, and s^n satisfy inequality (4.17) at equality. In particular, when $n_1 = n$, the points $p_1^n, \dots, p_n^n, q_n^n, r^n$, and s^n satisfy inequality (4.9) at equality.*

Proof. Substituting the coordinates of p_l^n , where $l \in \{1, \dots, n_1\}$, in the left-hand side of (4.17), we have

$$\begin{aligned}
& \alpha_l - 2\beta^{(n_1)} \prod_{k=l+1}^{n_1} \left\lfloor \frac{\beta^{(k-1)}}{\alpha_k} \right\rfloor - \sum_{i=l+1}^{n_1} \left(\alpha_i - 2\beta^{(n_1)} \prod_{k=i+1}^{n_1} \left\lfloor \frac{\beta^{(k-1)}}{\alpha_k} \right\rfloor \right) \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor + \beta^{(n_1)} \\
&= \alpha_l - 2\beta^{(n_1)} \prod_{k=l+1}^{n_1} \left\lfloor \frac{\beta^{(k-1)}}{\alpha_k} \right\rfloor - \sum_{i=l+1}^{n_1} \alpha_i \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor + 2\beta^{(n_1)} \sum_{i=l+1}^{n_1} \prod_{k=i+1}^{n_1} \left\lfloor \frac{\beta^{(k-1)}}{\alpha_k} \right\rfloor \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \\
&+ \beta^{(n_1)} \\
&= \alpha_l - 2\beta^{(n_1)} \prod_{k=l+1}^{n_1} \left\lfloor \frac{\beta^{(k-1)}}{\alpha_k} \right\rfloor - \sum_{i=l+1}^{n_1} \alpha_i \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor + 2\beta^{(n_1)} \left(\prod_{k=l+1}^{n_1} \left\lfloor \frac{\beta^{(k-1)}}{\alpha_k} \right\rfloor - 1 \right) + \beta^{(n_1)} \\
&= \alpha_l - \left(\beta^{(n_1)} + \sum_{i=l+1}^{n_1} \alpha_i \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) = \alpha_l - \beta^{(l)}
\end{aligned}$$

Notice that the second identity above is based on Lemma 36. The right-hand side of the last identity is the right-hand side of (4.17) for p_l^n . Therefore p_l^n satisfies (4.17) at equality. Substituting the coordinates of q_l^n , where $l \in \{n_1, \dots, n\}$, into (4.17), we get $\beta^{(n_1)} \leq \beta^{(l)} + \sum_{i=n_1+1}^l \alpha_i \lfloor \beta^{(i-1)} / \alpha_i \rfloor$, which holds at equality. Substituting the coordinates of r^n into (4.17), we get $\beta^{(n_1)} \leq \sum_{i=n_1+1}^n \alpha_i \lfloor \beta^{(i-1)} / \alpha_i \rfloor + \beta^{(n)}$, which holds at equality. Finally, s^n satisfies (4.17) at equality because p_1^n does so. This is true because the only difference between s^n and p_1^n is that in s^n , we have $w^- = \alpha_1 - \beta^{(1)}$

and $t = 0$, while in p_1^n , we have $w^- = 0$ and $t = \alpha_1 - \beta^{(1)}$. But this does not make any difference with respect to satisfying inequality (4.17). The second part of the lemma is just the special case of the first part when $n_1 = n$. \square

Theorem 44. *Let $n_1 \leq n$ and assume conditions $\alpha_i \lceil \beta^{(i-1)} / \alpha_i \rceil \leq \alpha_{i-1}$ for $i = 2, \dots, n_1$ hold. Then inequality (4.17) defines a facet for $\text{conv}(Q^n)$ if $\lfloor \beta^{(i-1)} / \alpha_i \rfloor \neq 0$, $i = n_1 + 1, \dots, n$. In particular, the n -step conic MIR inequality (4.9) defines a facet for $\text{conv}(Q^n)$ if conditions (4.1) are satisfied.*

Proof. The validity of (4.17) was proved in Corollary 40. Q^n is clearly full-dimensional as we can easily find a point $p \in Q^n$ such that $p + e_j \in Q^n$ for all unit vectors $e_j \in \mathbb{R}^{n+3}$. By Lemmas 42 and 43, the $n + 3$ points $p_1^n, \dots, p_{n_1}^n, q_{n_1}^n, \dots, q_n^n, r^n$ and s^n are all in Q^n and satisfy (4.17) at equality. It only remains to show that these points are affinely independent. Consider the $(n + 3) \times (n + 3)$ matrix whose rows from top to bottom correspond to the points $p_1^n, \dots, p_{n_1}^n, q_{n_1}^n, \dots, q_n^n, r^n, s^n$ and its columns are rearranged from left to right in the order $(t, y_1, \dots, y_n, w^+, w^-)$. We append a column of 1's to the left of this matrix to get the following $(n + 3) \times (n + 4)$ matrix:

$$\begin{array}{c} \begin{matrix} & 0 & 1 & 2 & 3 & \dots & n_1+1 & n_1+2 & \dots & n+1 & n+2 & n+3 \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ n_1 \\ n_1+1 \\ \vdots \\ n+1 \\ n+2 \\ n+3 \end{matrix} \begin{bmatrix} 1 & \alpha_1 - \beta^{(1)} & \left\lceil \frac{\beta}{\alpha_1} \right\rceil & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_{n_1} - \beta^{(n_1)} & \left\lceil \frac{\beta}{\alpha_1} \right\rceil & \left\lceil \frac{\beta^{(1)}}{\alpha_2} \right\rceil & \dots & \left\lceil \frac{\beta^{(n_1-1)}}{\alpha_{n_1}} \right\rceil & 0 & \dots & 0 & 0 & 0 \\ 1 & \beta^{(n_1)} & \left\lceil \frac{\beta}{\alpha_1} \right\rceil & \left\lceil \frac{\beta^{(1)}}{\alpha_2} \right\rceil & \dots & \left\lceil \frac{\beta^{(n_1-1)}}{\alpha_{n_1}} \right\rceil & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta^{(n)} & \left\lceil \frac{\beta}{\alpha_1} \right\rceil & \left\lceil \frac{\beta^{(1)}}{\alpha_2} \right\rceil & \dots & \left\lceil \frac{\beta^{(n_1-1)}}{\alpha_{n_1}} \right\rceil & \dots & \dots & \left\lceil \frac{\beta^{(n-1)}}{\alpha_n} \right\rceil & 0 & 0 \\ 1 & 0 & \left\lceil \frac{\beta}{\alpha_1} \right\rceil & \left\lceil \frac{\beta^{(1)}}{\alpha_2} \right\rceil & \dots & \left\lceil \frac{\beta^{(n_1-1)}}{\alpha_{n_1}} \right\rceil & \dots & \dots & \left\lceil \frac{\beta^{(n-1)}}{\alpha_n} \right\rceil & \beta^{(n)} & 0 \\ 1 & 0 & \left\lceil \frac{\beta}{\alpha_1} \right\rceil & 0 & \dots & \dots & \dots & \dots & 0 & 0 & \alpha_1 - \beta^{(1)} \end{bmatrix} \end{array}.$$

This result is partly due to Sina Masihabadi.

It is enough to show that the rows of this matrix are linearly independent. Denote the entry (i, j) of this matrix by h_{ij} , where $i \in \{1, \dots, n+3\}$ and $j = \{0, 1, \dots, n+3\}$ (the column of 1's is column 0). Notice that rows $n_1+2, \dots, n+3$ of the matrix are linearly independent. This is true because for $i = n_1+2, \dots, n+1$, we have $h_{ii} = \lfloor b^{(i-2)}/\alpha_{i-1} \rfloor$, which is non-zero by the assumption. Also $h_{n+2, n+2} = b^{(n)}$, which is non-zero by our general assumption, and $h_{n+3, n+3} = \alpha_1 - \beta^{(1)}$, which is non-zero by definition. Moreover, $h_{ij} = 0$ for $i = n_1+2, \dots, n+2; j > i$. Therefore rows $n_1+2, \dots, n+3$ are linearly independent. Now notice that $h_{ij} = 0$ for $i = 1, \dots, n_1+1; j = n_1+2, \dots, n+3$. Therefore rows $n_1+2, \dots, n+3$ are linearly independent from rows $1, \dots, n_1+1$. So it remains to show that the $(n_1+1) \times (n_1+2)$ sub-matrix formed by rows $1, \dots, n_1+1$ and columns $0, 1, \dots, n_1+1$ has linearly independent rows. Consider this sub-matrix and perform the following set of row operations on it: Starting with $i = n_1+1$ down to 2, add -1 times row $i-1$ to row i . The result will be

$$\begin{array}{c} \begin{matrix} 1 \\ 2 \\ \vdots \\ n_1 \\ n_1+1 \end{matrix} \end{array} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & n_1-1 & n_1 & n_1+1 \\ 1 & \alpha_1 - \beta^{(1)} & \left\lfloor \frac{\beta}{\alpha_1} \right\rfloor & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_2 - \beta^{(2)} - \alpha_1 + \beta^{(1)} & -1 & \left\lfloor \frac{\beta^{(1)}}{\alpha_2} \right\rfloor & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_{n_1} - \beta^{(n_1)} - \alpha_{n_1-1} + \beta^{(n_1-1)} & 0 & 0 & 0 & \cdots & 0 & -1 & \left\lfloor \frac{\beta^{(n_1-1)}}{\alpha_{n_1}} \right\rfloor \\ 0 & -\alpha_{n_1} + 2\beta^{(n_1)} & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}.$$

Now in the matrix above, starting with $i = n_1+1$ down to 2, update row $i-1$ by adding $\lfloor \beta^{(i-2)}/\alpha_{i-1} \rfloor$ times updated row i to it. We will get

$$\begin{array}{c} \begin{matrix} 1 \\ 2 \\ \vdots \\ n_1 \\ n_1+1 \end{matrix} \end{array} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & n_1-1 & n_1 & n_1+1 \\ 1 & -\beta + 2\beta^{(n_1)} \prod_{i=1}^{n_1} \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\alpha_1 + 2\beta^{(n_1)} \prod_{i=2}^{n_1} \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\alpha_{n_1-1} + 2\beta^{(n_1)} \left\lfloor \frac{\beta^{(n_1-1)}}{\alpha_{n_1}} \right\rfloor & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & -\alpha_{n_1} + 2\beta^{(n_1)} & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}.$$

Notice that columns $0, 2, \dots, n_1+1$ of the matrix above form a lower-triangular

matrix that all its diagonal entries are non-zero. This simply implies that the rows of the above matrix are linearly independent and completes the proof. Notice that the second part of the theorem is the special case of $n_1 = n$ as in this case inequality (4.17) becomes the same as (4.9). We see that in this case the only conditions required are conditions (4.1). \square

IV.3 Polyhedral Second-order Conic Inequalities

Notice that the n -step conic MIR inequality presented in Sect. IV.2 is a linear inequality. In [19], in addition to showing the validity if the 1-step conic MIR inequality (4.3) (with $\alpha_1 = 1$), i.e.

$$(1 - 2(\beta - \lfloor \beta \rfloor))(y_1 - \lfloor \beta \rfloor) - (\beta - \lfloor \beta \rfloor) \leq t + w^+ + w^-, \quad (4.18)$$

for \overline{Q} , Atamtürk and Narayanan observed that this inequality remains valid if its left-hand side is multiplied by -1 , and as a result the PSOC-form inequality

$$\left| (1 - 2(\beta - \lfloor \beta \rfloor))(y_1 - \lfloor \beta \rfloor) - (\beta - \lfloor \beta \rfloor) \right| \leq t + w^+ + w^- \quad (4.19)$$

is also valid for \overline{Q} . Clearly, (4.18) is a relaxation of (4.19). In addition to having a stronger linear relaxation, PSOC inequalities such as (4.19) are of interest because they can be used to define nonlinear inequalities as explained in Section II.5, [19]. The set Q_0^n has a form like constraint (2.31). So an interesting question is that will the stronger inequality

$$\left| \sum_{i=1}^n \left(\alpha_i - 2\beta^{(n)} \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) + \beta^{(n)} \right| \leq t, \quad (4.20)$$

which is obtained by taking absolute value of the left-hand side of the n -step conic MIR inequality for Q_0^n and is of the useful PSOC form (2.34), valid for Q_0^n for $n > 0$?

Or equivalently (based on Lemma 34), is the inequality

$$\left| \sum_{i=1}^n \left(\alpha_i - 2\beta^{(n)} \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) + \beta^{(n)} \right| \leq t + w^+ + w^- \quad (4.21)$$

valid for Q^n for $n > 1$? In Theorem 46, we will show that under conditions (4.1), inequality (4.21) is valid for Q^n (or (4.20) is valid for Q_0^n) if and only if conditions (4.1) are satisfied at equality.

First, we prove the validity of (4.21) when $n = 1$ for Q^1 , which is needed in proving Theorem 46. For $\alpha_1 = 1$, this validity, or in other words the validity of (4.19) for \overline{Q} , is stated, but not proved, in [19].

Lemma 45. *The inequality*

$$\left| (\alpha_1 - 2\beta^{(1)}) (y_1 - \lfloor \beta/\alpha_1 \rfloor) + \beta^{(1)} \right| \leq t + w^+ + w^- \quad (4.22)$$

is valid for Q^1 .

Proof. By Lemma 38, the 1-step conic MIR inequality (4.3) is valid for Q^1 . Therefore using Lemma 34, it remains to show that

$$-(\alpha_1 - 2\beta^{(1)}) (y_1 - \lfloor \beta/\alpha_1 \rfloor) - \beta^{(1)} \leq t \quad (4.23)$$

is valid for Q_0^1 . Let $(y_1, t) \in Q_0^1$. We consider two cases: First, assume $\alpha_1 y_1 - \beta > 0$, which since $y_1 \in \mathbb{Z}$, means

$$y_1 - \lfloor \beta/\alpha_1 \rfloor \geq 1 \quad (4.24)$$

(recall the general assumption that $b/\alpha_1 \notin \mathbb{Z}$). Based on the defining inequality of Q_0^1 , in this case we have $\alpha_1 y_1 - \beta \leq t$. This can be written as

$$\alpha_1 (y_1 - \lfloor \beta/\alpha_1 \rfloor) - \beta^{(1)} \leq t. \quad (4.25)$$

Multiplying (4.24) by $-2(\alpha_1 - \beta^{(1)})$ and adding it to (4.25) we get

$$-(\alpha_1 - 2\beta^{(1)})(y_1 - \lfloor \beta/\alpha_1 \rfloor) - \beta^{(1)} + 2(\alpha_1 - \beta^{(1)}) \leq t.$$

Clearly this inequality dominates (4.23) because $\alpha_1 - \beta^{(1)} > 0$. Therefore (4.23) is also satisfied. Next assume $\alpha_1 y_1 - \beta < 0$, which since $y_1 \in \mathbb{Z}$, means

$$y_1 - \lfloor \beta/\alpha_1 \rfloor \leq 0. \quad (4.26)$$

Based on the defining inequality of Q_0^1 , in this case we have $-\alpha_1 y_1 + \beta \leq t$. This can be written as

$$-\alpha_1 (y_1 - \lfloor \beta/\alpha_1 \rfloor) + \beta^{(1)} \leq t. \quad (4.27)$$

Multiplying (4.26) by $2\beta^{(1)}$, adding it to (4.27) we get

$$-(\alpha_1 - 2\beta^{(1)})(y_1 - \lfloor \beta/\alpha_1 \rfloor) + \beta^{(1)} \leq t.$$

As $\beta^{(1)} > 0$, this inequality also dominates (4.23) so again (4.23) is satisfied. This completes the proof. \square

Theorem 46. *Assume conditions (4.1) hold. Inequality (4.21) is valid for Q^n if and only if conditions (4.1) are satisfied at equality, i.e.*

$$\alpha_{i-1} = \alpha_i \lceil \beta^{(i-1)}/\alpha_i \rceil \quad \text{for } i = 2, \dots, n. \quad (4.28)$$

Proof. First, we prove the sufficiency of (4.28). Assume (4.28) holds. Based on Lemma 34, we prove the validity of (4.21) for Q^n by proving the validity of (4.20) for Q_0^n . According to (4.28), we have

$$\alpha_i = \alpha_n \prod_{k=i+1}^n \lceil \beta^{(k-1)}/\alpha_k \rceil, \quad \text{for } i = 1, \dots, n. \quad (4.29)$$

Replacing for α_i , $i = 1, \dots, n$, in (4.20) from (4.29), inequality (4.20) reduces to

$$\left| (\alpha_n - 2\beta^{(n)}) \left(\sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil y_i - \sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) + \beta^{(n)} \right| \leq t. \quad (4.30)$$

To see the validity of (4.30) for Q_0^n , observe that using (4.29), the defining inequality of Q_0^n , i.e. $\left| \sum_{i=1}^n \alpha_i y_i - \beta \right| \leq t$, can be written as

$$\left| \alpha_n \sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil y_i - \beta \right| \leq t. \quad (4.31)$$

If we treat $\sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \beta^{(k-1)}/\alpha_k \right\rceil y_i$ in (4.31) as the integer variable y_1 in Q_0^1 , and α_n in (4.31) as α_1 in Q_0^1 . Based on Lemmas 34 and 45, inequality (4.22) written for (4.31), i.e.

$$\left| (\alpha_n - 2\beta^{(\alpha_n)}) \left(\sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil y_i - \left\lfloor \frac{\beta}{\alpha_n} \right\rfloor \right) + \beta^{(\alpha_n)} \right| \leq t, \quad (4.32)$$

will be valid for Q_0^n , where we define $\beta^{(\alpha_n)} := \beta - \alpha_n \lfloor \beta/\alpha_n \rfloor$. Now (4.32) is the same as (4.30) because of identities $\sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor = \left\lfloor \frac{\beta}{\alpha_n} \right\rfloor$ and $\beta^{(\alpha_n)} = b^{(n)}$, the validity of which can be shown as follows:

$$\begin{aligned} \sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \frac{\beta^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor &= \sum_{i=1}^n \frac{\alpha_i}{\alpha_n} \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \\ &= \left\lfloor \frac{b^{(n-1)}}{\alpha_n} \right\rfloor + \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \\ &= \left\lfloor \frac{\beta^{(n-1)}}{\alpha_n} + \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right\rfloor \\ &= \left\lfloor \frac{1}{\alpha_n} \left(\beta^{(n-1)} + \sum_{i=1}^{n-1} \alpha_i \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) \right\rfloor \\ &= \left\lfloor \frac{b}{\alpha_n} \right\rfloor. \end{aligned}$$

The first identity above is true because of (4.29) and the third identity is true because $\sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor$ is an integer. From identities above we also have

$$\left\lfloor \frac{b}{\alpha_n} \right\rfloor = \sum_{i=1}^n \frac{\alpha_i}{\alpha_n} \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor. \quad (4.33)$$

Therefore

$$\beta^{(\alpha_n)} = \beta - \alpha_n \left\lfloor \beta / \alpha_n \right\rfloor = \beta - \sum_{i=1}^n \alpha_i \left\lfloor \beta^{(i-1)} / \alpha_i \right\rfloor = \beta^{(n)},$$

where the second identity is based on (4.33).

Next, we prove the necessity of (4.28). Based on Lemma 34, it is enough to show that (4.20) is valid for Q_0^n only if (4.28) holds. Assume (4.20) is valid for Q_0^n . Therefore

$$-\sum_{i=1}^n \left(\alpha_i - 2\beta^{(n)} \prod_{k=i+1}^n \left\lfloor \frac{\beta^{(k-1)}}{\alpha_k} \right\rfloor \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) - \beta^{(n)} \leq t \quad (4.34)$$

is valid for Q_0^n . By Lemma 35, this in turn is equivalent to the validity of

$$-\sum_{i=1}^n \left(\alpha_i - 2\beta^{(n)} \prod_{k=i+1}^n \left\lfloor \frac{\beta^{(k-1)}}{\alpha_k} \right\rfloor \right) \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) - \beta^{(n)} \leq \left| \sum_{i=1}^n \alpha_i y_i - \beta \right|. \quad (4.35)$$

for Q_0^n . Consider a subset of Q_0^n defined as $Q_0^{n-} := Q_0^n \cap \{(y, t) \in \mathbb{R}^{n+1} : \sum_{i=1}^n \alpha_i y_i - \beta \leq 0\}$. Since (4.35) is valid for Q_0^n , it is also valid for Q_0^{n-} . However for Q_0^{n-} , the right-hand side of (4.35) can be replaced with $\beta - \sum_{i=1}^n \alpha_i y_i = -\sum_{i=1}^n \alpha_i (y_i - \left\lfloor \beta^{(i-1)} / \alpha_i \right\rfloor) + \beta^{(n)}$.

This simplifies (4.35) to

$$\sum_{i=1}^n \prod_{k=i+1}^n \left\lfloor \frac{\beta^{(k-1)}}{\alpha_k} \right\rfloor \left(y_i - \left\lfloor \frac{\beta^{(i-1)}}{\alpha_i} \right\rfloor \right) \leq 1. \quad (4.36)$$

For simplicity of notation define

$$\lambda_i := \prod_{k=i+1}^n \left\lfloor \beta^{(k-1)} / \alpha_k \right\rfloor, \quad i = 1, \dots, n. \quad (4.37)$$

Notice that $\lambda_i > 0$, $i = 1, \dots, n$. Based on (4.36) and (4.37), inequality

$$\sum_{i=1}^n \lambda_i (y_i - \lfloor \beta^{(i-1)} / \alpha_i \rfloor) \leq 1 \quad (4.38)$$

is valid for Q_0^{n-} . We show that this implies

$$\lambda_{i-1} / \alpha_{i-1} = \lambda_i / \alpha_i, \quad i = 2, \dots, n. \quad (4.39)$$

Note that this will complete the proof because by replacing for λ_{i-1} and λ_i in (4.39) from (4.37) and simplifying, it is easy to see that (4.39) is equivalent to conditions (4.28). To see that (4.39) holds, first note that

$$\lambda_{i-1} / \alpha_{i-1} \leq \lambda_i / \alpha_i, \quad i = 2, \dots, n, \quad (4.40)$$

because of conditions (4.1). This again can be easily verified by replacing for λ_{i-1} and λ_i in (4.40) from (4.37). By contradiction assume (4.39) does not hold. Based on (4.40), this means there exists $l \in \{2, \dots, n\}$ such that

$$\lambda_1 / \alpha_1 < \lambda_l / \alpha_l. \quad (4.41)$$

We find a point $(\bar{y}, \bar{t}) \in Q_0^{n-}$ that violates (4.38) contradicting the validity of (4.38) for Q_0^{n-} . Set $\bar{y}_i = \lfloor \beta^{(i-1)} / \alpha_i \rfloor$ for all $i \in \{2, \dots, n\} \setminus \{l\}$. For (\bar{y}, \bar{t}) to belong to Q_0^{n-} , it should satisfy $\sum_{i=1}^n \alpha_i y_i \leq \beta$. This means we must have

$$\bar{y}_1 \leq \lfloor \beta / \alpha_1 \rfloor + \beta^{(n)} / \alpha_1 - (\alpha_l / \alpha_1) (\bar{y}_l - \lfloor \beta^{(l-1)} / \alpha_l \rfloor). \quad (4.42)$$

For (\bar{y}, \bar{t}) to violate (4.38), we must have

$$\bar{y}_1 > \lfloor \beta / \alpha_1 \rfloor + 1 / \lambda_1 - (\lambda_l / \lambda_1) (\bar{y}_l - \lfloor \beta^{(l-1)} / \alpha_l \rfloor). \quad (4.43)$$

The difference between the left-hand side of (4.42) and the left-hand side of (4.43) is

$$\beta^{(n)}/\alpha_1 - 1/\lambda_1 + (\lambda_l/\lambda_1 - \alpha_l/\alpha_1) (\bar{y}_l - \lfloor \beta^{(l-1)}/\alpha_l \rfloor) \quad (4.44)$$

Since $\alpha_l, \lambda_1 > 0$, inequality (4.41) implies $\lambda_l/\lambda_1 - \alpha_l/\alpha_1 > 0$. This means the coefficient of \bar{y}_l in (4.44) is positive. Hence, by increasing \bar{y}_l , (4.44) can be made as large as desired. Set \bar{y}_l to a non-negative integer for which (4.44) is greater than 1. This guarantees there exists an integer value for \bar{y}_1 that satisfies (4.42) and (4.43). Set \bar{y}_1 to this value and $\bar{t} = \left\lceil \sum_{i=1}^n \alpha_i \bar{y}_i - \beta \right\rceil$. As a result $(\bar{y}, \bar{t}) \in Q_0^{n-}$ and violates (4.38), which is the contradiction we were looking for. This completes the proof. \square

Theorem 46 shows that the n -step conic MIR can be employed in generating a nonlinear inequality of the form (2.35) as explained at the beginning of this section only if conditions (4.1) are satisfied at equality (of course for $n = 1$ no condition exists).

IV.4 n -step Conic MIR Inequality for the General PSOC Set S

In this section, we show that n -step conic MIR facet of Q^n can be used to generate a n -step conic MIR inequality for the general PSOC set

$$S = \left\{ (x, z^+, z^-, t) \in \mathbb{Z}_+^N \times \mathbb{R}_+^3 : \left| \sum_{j \in J} a_j x_j + z^+ - z^- - b \right| \leq t \right\},$$

where $J = \{1, \dots, N\}$. Atamtürk and Narayanan [19] showed that it is enough to study the facets of $\text{conv}(S)$ to derive facets for the convex hull of a set like $\bar{S} = \{(x, z, t) \in \mathbb{Z}_+^N \times \mathbb{R}_+^{L+1} : |ax + gz - b| \leq t\}$ because the coefficients of continuous variables z in any facet for $\text{conv}(\bar{S})$ are proportional to the absolute values of coefficients g .

We will show that like n -step MIR inequality of Kianfar and Fathi [78], which was generated by the n -step MIR function $\mu_{\alpha,b}^n$, the n -step conic MIR inequality is generated by a function which we refer to as the n -step conic MIR function. We first define the

n -step conic MIR function $\phi_{\alpha,b}^n$, and then present the n -step conic MIR inequality and prove its validity.

Definition 47. *The n -step conic MIR function for the parameter vector $\alpha = (\alpha_1, \dots, \alpha_n)$ and the right-hand side b is defined as*

$$\phi_{\alpha,b}^n(u) = u - 2\mu_{\alpha,b}^n(u). \quad (4.45)$$

Theorem 48. *Given a parameter vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha > 0$, if conditions (4.1) hold, the n -step conic MIR inequality*

$$\sum_{j \in J} \phi_{\alpha,b}^n(a_j)x_j - \phi_{\alpha,b}^n(b) \leq t + z^+ + z^- \quad (4.46)$$

is valid for S .

Proof. We partition the set J into $n+1$ sets J_0, J_1, \dots, J_n . The defining inequality of S can then be written as

$$\left| \sum_{m=0}^n \sum_{j \in J_m} a_j x_j + z^+ - z^- - b \right| \leq t. \quad (4.47)$$

For $j \in J_m$, replace a_j in (4.47) from the following identities

$$a_j = \begin{cases} \sum_{i=1}^m \alpha_i \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor + \alpha_{m+1} \left\lfloor \frac{a_j^{(m)}}{\alpha_{m+1}} \right\rfloor - (\alpha_{m+1} - a_j^{(m+1)}) & j \in J_m; m = 0, \dots, n-1 \\ \sum_{i=1}^n \alpha_i \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor + a_j^{(n)} & j \in J_n. \end{cases}$$

to get

$$\begin{aligned} & \left| \sum_{m=0}^{n-1} \sum_{j \in J_m} \left(\sum_{i=1}^m \alpha_i \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor + \alpha_{m+1} \left\lfloor \frac{a_j^{(m)}}{\alpha_{m+1}} \right\rfloor - (\alpha_{m+1} - a_j^{(m+1)}) \right) x_j \right. \\ & \quad \left. + \sum_{j \in J_n} \left(\sum_{i=1}^n \alpha_i \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor + a_j^{(n)} \right) x_j + z^+ - z^- - b \right| \leq t. \quad (4.48) \end{aligned}$$

After rearranging the terms in (4.48), we get

$$\left| \sum_{i=1}^n \alpha_i \left(\sum_{j \in J_{i-1}} \left\lceil \frac{a_j^{(i-1)}}{\alpha_i} \right\rceil x_j + \sum_{m=i}^n \sum_{j \in J_m} \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor x_j \right) + \left(\sum_{j \in J_n} a_j^{(n)} x_j + z^+ \right) - \left(\sum_{m=0}^{n-1} \sum_{j \in J_m} (\alpha_{m+1} - a_j^{(m+1)}) x_j + z^- \right) - b \right| \leq t \quad (4.49)$$

Now compare (4.49) with Q^n . For $i = 1, 2, \dots, n$, the expression multiplied by α_i in the first summation is an integer, which is also nonnegative for $i = 2, \dots, n$. So it can be treated as y_i in Q^n . The expressions in the second and third parentheses are nonnegative real values and can be treated as z^+ and z^- in Q^n , respectively. Since conditions (4.1) hold, by Theorem 39, the n -step conic MIR inequality (4.9) is valid for Q^n . Replacing y_1, \dots, y_n, z^+ , and z^- in (4.9) with their respective expressions, we get the following valid inequality for S :

$$\begin{aligned} & \sum_{i=1}^n \left(\alpha_i - 2b^{(n)} \prod_{k=i+1}^n \left\lceil \frac{b^{(k-1)}}{\alpha_k} \right\rceil \right) \left(\sum_{j \in J_{i-1}} \left\lceil \frac{a_j^{(i-1)}}{\alpha_i} \right\rceil x_j \right. \\ & \quad \left. + \sum_{m=i}^n \sum_{j \in J_m} \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor x_j - \left\lfloor \frac{b^{(i-1)}}{\alpha_i} \right\rfloor \right) + b^{(n)} \\ & \leq t + \sum_{j \in J_n} a_j^{(n)} x_j + z^+ + \sum_{m=0}^{n-1} \sum_{j \in J_m} (\alpha_{m+1} - a_j^{(m+1)}) x_j + z^- \quad (4.50) \end{aligned}$$

After rearranging the terms and using Lemma 36 on the constant term, inequality (4.50) can be written as

$$\sum_{j \in J} f(a_j) x_j - \left(b - 2b^{(n)} \prod_{k=1}^n \left\lceil b^{(k-1)} / \alpha_k \right\rceil \right) \leq t + z^+ + z^-, \quad (4.51)$$

where

$$f(a_j) = \begin{cases} a_j - 2 \left(b^{(n)} \sum_{i=1}^m \prod_{k=i+1}^n \left\lceil \frac{b^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor + b^{(n)} \prod_{k=m+2}^n \left\lceil \frac{b^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{a_j^{(m)}}{\alpha_{m+1}} \right\rfloor \right) & j \in J_m; m = 0, \dots, n-1 \\ a_j - 2 \left(b^{(n)} \sum_{i=1}^n \prod_{k=i+1}^n \left\lceil \frac{b^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{a_j^{(i-1)}}{\alpha_i} \right\rfloor + a_j^{(n)} \right) & j \in J_n \end{cases}$$

For $m = 0, 1, \dots, n$, denote the function $f(a_j)$ for $j \in J_m$ by $f_m(a_j)$ (in other words $f(a_j) = f_m(a_j)$ if $j \in J_m$). To get the strongest inequality (4.51), J should be partitioned such that for $m = 0, 1, \dots, n$, we have $f_m(a_j) = \max_{k \in \{0, 1, \dots, n\}} f_k(a_j)$ for $j \in J_m$. For $m = 0, 1, \dots, n$, denote the n -step MIR function $\mu_{\alpha, b}^n(a_j)$ for I_n^m by $\mu_m(a_j)$ (refer to Sect. II.5). Examining the above formulation for $f_m(a_j)$, we see that

$$f_m(a_j) = a_j - 2\mu_m(a_j), \quad m = 0, 1, \dots, n. \quad (4.52)$$

According to [78], if $a_j \in I_m^n$, then $\mu_m(a_j) = \min_{k \in \{0, 1, \dots, n\}} \mu_k(a_j)$. Therefore, based on (4.52), the strongest inequality is obtained if we partition J as follows: $J_m = \{j : a_j \in I_m^n\}$, $m = 0, 1, \dots, n$. By this partitioning, we will have $f(a_j) = \phi_{\alpha, b}^n(a_j)$. Also notice that $b \in I_0^n$, and hence $\phi_{\alpha, b}^n(b) = b - 2b^{(n)} \prod_{k=1}^n \lceil b^{(k-1)} / \alpha_k \rceil$. Therefore, inequality (4.51) becomes the same as inequality (4.46). This completes the proof. \square

Remark 4. If for some variable x_k , the coefficient a_k in S is an integer multiple of α_1 , then the n -step conic MIR inequality (4.46) will be valid even for the relaxation of S in which x_k is not necessarily nonnegative. This is easy to see by examining (4.49). The only place that x_k will appear will be inside the parentheses that is multiplied by α_1 , which represents y_1 in Q^n . Since y_1 is unrestricted in Q^n , x_k can also be unrestricted without distorting the proof of Theorem 48.

Lemma 49. *The n -step conic MIR function $\phi_{\alpha,b}^n(u)$ is piecewise linear, continuous, superadditive, and has two slopes, i.e. 1 and -1 . Moreover, we have*

$$\phi_{\lambda\alpha,\lambda b}^n(\lambda u) = \lambda \phi_{\alpha,b}^n(u). \quad (4.53)$$

Proof. As mentioned in Sect. 2, it was shown in [18, 78] that the n -step MIR function $\mu_{\alpha,b}^n(u)$ is piecewise linear, continuous, subadditive, and has two slopes, i.e. 0 and 1. From (4.45), it immediately follows that $\phi_{\alpha,b}^n(u)$ is piecewise linear, continuous, and has two slopes, i.e. 1 and -1 . Note that it is also superadditive because u and $-\mu_{\alpha,b}^n(u)$ are both superadditive so any nonnegative linear combination of them is also superadditive. Identity (4.53) can be easily verified based on (4.45) and the definition of $\mu_{\alpha,b}^n(u)$. \square

Figures 2 to 5 show examples of 1, 2, 3, and 4-step conic MIR functions constructed for $b = 0.8$ and the parameter vector α given in each figure. Note that based on (4.53), these graphs can be scaled without any change in their shape.

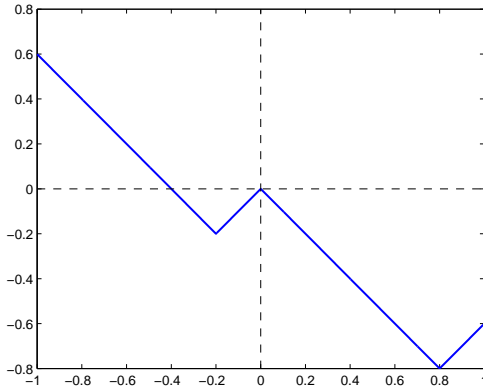


Fig. 2. $\phi_{\alpha,0.8}^1(u), \alpha = 1$

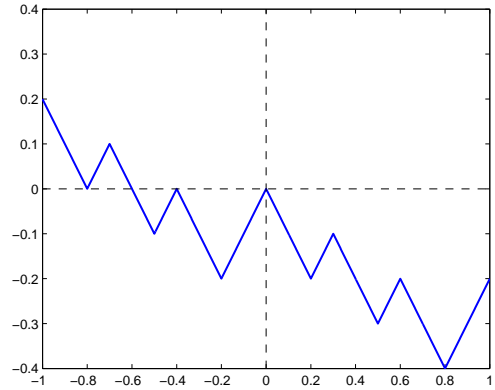
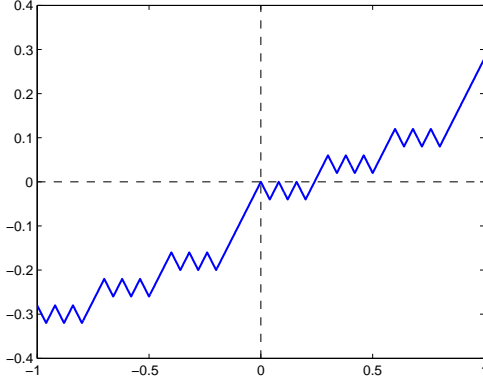
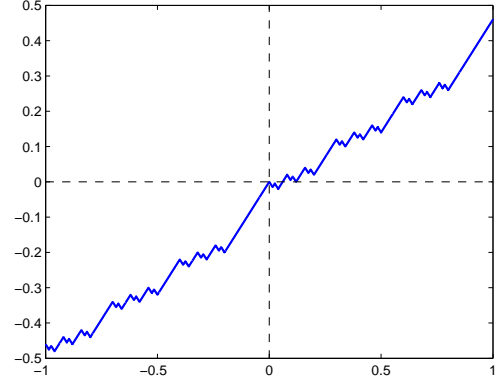


Fig. 3. $\phi_{\alpha,0.8}^2(u), \alpha = (1, 0.3)$

Fig. 4. $\phi_{\alpha,0.8}^3(u), \alpha = (1, 0.3, 0.08)$ Fig. 5. $\phi_{\alpha,0.8}^4(u), \alpha = (1, 0.3, 0.08, 0.025)$

Interestingly, the n -step conic MIR inequality is facet-defining for the $\text{conv}(S)$ in many cases:

Theorem 50. *The n -step conic MIR inequality (4.46) defines a facet for $\text{conv}(S)$ if the following conditions are satisfied:*

- i.* $\alpha_k = a_{j_k}$, where $j_k \in J$ and $a_{j_k} > 0$ for $k = 1, \dots, n$.
- ii.* $\alpha_{k-1} \geq \alpha_k \lceil b^{(k-1)} / \alpha_k \rceil$ for $k = 2, \dots, n$.
- iii.* $b > 0$, $b^{(n)} > 0$, and $\lfloor a_j / \alpha_1 \rfloor \leq \lfloor b / \alpha_1 \rfloor$ for all $j \in J \setminus J_\alpha$.

Proof. Conditions *i* and *ii* imply conditions (4.1) and hence inequality (4.46) is valid for S based on Theorem 48. The set S is clearly full-dimensional as we can easily find a point $p \in S$ such that $p + e_j \in S$ for all unit vectors $e_j \in \mathbb{R}^{N+3}$. Let J_α be the set of indices of coefficients that are chosen as parameters $\alpha_1, \dots, \alpha_n$, i.e. $J_\alpha := \{a_{j_1}, \dots, a_{j_n}\}$. We now list $N + 3$ affinely independent points in S that lie on the face defined by inequality (4.46). For each point we only specify the non-zero x components as well as z^+ , z^- , and t :

- The point $P_0^0 = (x, z^+, z^-, t)$ such that $x_{j_k} = \left\lfloor \frac{b^{(k-1)}}{\alpha_k} \right\rfloor$ for $k = 1, \dots, n$, $z^+ = 0$, $z^- = 0$, and $t = b^{(n)}$.
- The point $P_0^1 = (x, z^+, z^-, t)$ such that $x_{j_k} = \left\lfloor \frac{b^{(k-1)}}{\alpha_k} \right\rfloor$ for $k = 1, \dots, n$, $z^+ = b^{(n)}$, $z^- = 0$ and $t = 0$.
- The point $P_0^2 = (x, z^+, z^-, t)$ such that $x_{j_1} = \left\lfloor \frac{b}{\alpha_1} \right\rfloor$, $z^+ = 0$, $z^- = \alpha_1 - b^{(1)}$, and $t = 0$.
- For each $j_k \in J_\alpha$ where $k = 1, \dots, n$, the point $P_{j_k} = (x, z^+, z^-, t)$ such that $x_{j_l} = \left\lfloor \frac{b^{(l-1)}}{\alpha_l} \right\rfloor$ for $l = 1, \dots, k-1$, $x_{j_k} = \left\lfloor \frac{b^{(k-1)}}{\alpha_k} \right\rfloor$, $z^+ = 0$, $z^- = 0$, and $t = \alpha_k - b^{(k)}$.
- For each $j \in J \setminus J_\alpha$ where $a_j \in I_m^n$ for $m = 0, \dots, n-1$, the point $P_j = (x, z^+, z^-, t)$ such that $x_{j_k} = \left\lfloor \frac{b^{(k-1)}}{\alpha_k} \right\rfloor - \left\lfloor \frac{a_j^{(k-1)}}{\alpha_k} \right\rfloor$ for $k = 1, \dots, m+1$, $x_j = 1$, $z^+ = 0$, $z^- = 0$, and $t = a_j^{(m+1)} - b^{(m+1)}$.
- For each $j \in J \setminus J_\alpha$ where $a_j \in I_n^n$, the point $P_j = (x, z^+, z^-, t)$ such that $x_{j_k} = \left\lfloor \frac{b^{(k-1)}}{\alpha_k} \right\rfloor - \left\lfloor \frac{a_j^{(k-1)}}{\alpha_k} \right\rfloor$ for $k = 1, \dots, n$, $x_j = 1$, $z^+ = 0$, $z^- = 0$, and $t = b^{(n)} - a_j^{(n)}$.

Given conditions *i*, *ii*, and *iii*, it is easy to verify that all these $N+3$ points belong to S and satisfy (4.46) at equality. To see that they are also affinely independent, consider the points $P_{j_k}, k = 1, \dots, n, P_0^0, P_0^1$, and P_0^2 . Note that if the coordinates of these points are put in the order $x_{j_k}, k = 1, \dots, n, z^+, z^-, t, x_j, j \in J \setminus J_\alpha$, these points are the same as $(p_1^n, 0), \dots, (p_n^n, 0), (q_n^n, 0), (r^n, 0)$ and $(s^n, 0)$, respectively, where $p_1^n, \dots, p_n^n, q_n^n, r^n$ and s^n are as defined in Definition 41. The proof of Theorem 44 for $n_1 = n$ showed that $p_1^n, \dots, p_n^n, q_n^n, r^n, s^n$ are affinely independent. Therefore $P_{j_k}, k = 1, \dots, n, P_0^0, P_0^1, P_0^2$ are also affinely independent. Moreover, notice that for each $j \in J \setminus J_\alpha$, we have $x_j = 1$ for the point P_j and $x_j = 0$ for all other $N+2$ points listed above. This implies that if the points $P_j, j \in J \setminus J_\alpha$ are also included, the resulting set of points, i.e. $P_j, j \in J, P_0^0, P_0^1, P_0^2$ remains affinely independent. This completes the proof. \square

Theorem 50 can also be written for the case $b < 0$:

Corollary 51. *The n -step conic MIR inequality*

$$\sum_{j \in J} \phi_{\alpha, -b}(-a_j)x_j - \phi_{\alpha, -b}(b) \leq t + z^+ + z^- \quad (4.54)$$

defines a facet for $\text{conv}(S)$ if the following conditions are satisfied:

- i. $\alpha_k = -a_{j_k}$, where $j_k \in J$ and $a_{j_k} < 0$ for $k = 1, \dots, n$.
- ii. $\alpha_{k-1} \geq \alpha_k \left\lceil (-b)^{(k-1)} / \alpha_k \right\rceil$ for $k = 2, \dots, n$.
- iii. $b < 0$, $(-b)^{(n)} > 0$, and $\lfloor -a_j / \alpha_1 \rfloor \leq \lfloor -b / \alpha_1 \rfloor$ for all $j \in J \setminus J_\alpha$.

Proof. We can also write the defining inequality of S as

$$\left| \sum_{j \in J} (-a_j)x_j - z^+ + z^- - (-b) \right| \leq t. \quad (4.55)$$

Now this corollary is directly implied by Theorem 50 written for (4.55). \square

Example 3. Consider the set

$$S = \left\{ (x, z^+, z^-, t) \in \mathbb{Z}_+^6 \times \mathbb{R}_+^3 : \left| 15x_1 + 6x_2 + 3x_3 - x_4 - 17x_5 + 16x_6 + z^+ - z^- - 25 \right| \leq t \right\}.$$

To generate a 3-step conic MIR inequality for this set choose $\alpha_1 = a_1 = 15$, $\alpha_2 = a_2 = 6$, and $\alpha_3 = a_3 = 3$. We have $b = 25$. Therefore $b^{(1)} = 10$, $b^{(2)} = 4$, $b^{(3)} = 1$, and $\lceil b / \alpha_1 \rceil = \lceil b^{(1)} / \alpha_2 \rceil = \lceil b^{(2)} / \alpha_3 \rceil = 2$. It can be easily verified that all the conditions of Theorem 50 are satisfied, and therefore the 3-step conic MIR inequality

$$7x_1 + 2x_2 + x_3 - x_4 - 9x_5 + 6x_6 - 9 \leq t + z^+ + z^-, \quad (4.56)$$

is valid and facet-defining for $\text{conv}(S)$. For the facet (4.56), the affinely independent points listed in the proof of Theorem 50 that belong to S and lie on the facet are the following 9 points:

$$\begin{array}{c}
P_1 \\
P_2 \\
P_3 \\
P_0^0 \\
P_0^1 \\
P_0^2 \\
P_4 \\
P_5 \\
P_6
\end{array}
\begin{bmatrix}
t & x_1 & x_2 & x_3 & z^+ & z^- & x_4 & x_5 & x_6 \\
5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
4 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
3 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Of course all the conditions of Theorem 50 are satisfied for 2-step and 1-step conic MIR too. The 2-step conic MIR inequality for S with $(\alpha_1, \alpha_2) = (15, 6)$ is

$$-x_1 - 2x_2 - 3x_3 - x_4 - x_5 - 2x_6 + 7 \leq t + z^+ + z^-, \quad (4.57)$$

and the 1-step conic MIR inequality for S with $\alpha_1 = 15$ is

$$-5x_1 - 6x_2 - 3x_3 - x_4 + 15x_5 - 6x_6 + 15 \leq t + z^+ + z^-. \quad (4.58)$$

Both (4.57) and (4.58) also define facets for $\text{conv}(S)$ based on Theorem 50.

IV.5 n -step Conic MIR dominates n -step MIR for S .

The defining inequality of the set S is equivalent to two linear inequalities:

$$\sum_{j \in J} a_j x_j + z^+ - z^- - t \leq b, \quad (4.59)$$

$$\sum_{j \in J} a_j x_j + z^+ - z^- + t \geq b. \quad (4.60)$$

In this section, we show that the n -step conic MIR inequality for the set S strictly dominates the n -step MIR inequalities written based on (4.59) and (4.60).

To get the n -step MIR inequality based on (4.59), we relax it to $\sum_{j \in J} a_j x_j - z^- - t \leq b$. According to (2.11) and Lemma 37, we get the following n -step MIR inequality for

S :

$$\sum_{j \in J} (a_j - \mu_{\alpha,b}^n(a_j)) x_j - z^- - t \leq b - \mu_{\alpha,b}^n(b). \quad (4.61)$$

To get the n -step MIR inequality based on (4.60), we relax it to $\sum_{j \in J} a_j x_j + z^+ + t \geq b$. According to (2.11), we get the following n -step MIR inequality for S :

$$\sum_{j \in J} \mu_{\alpha,b}^n(a_j) x_j + z^+ + t \geq \mu_{\alpha,b}^n(b). \quad (4.62)$$

First, we show that the n -step MIR inequalities (4.61) and (4.62) are dominated (implied) by the n -step conic MIR inequality (4.46). Let $\text{relax}(S)$ denote the set obtained from S by relaxing the integrality constraints.

Theorem 52. *If a point in $\text{relax}(S)$ satisfies the n -step conic MIR inequality (4.46), then it also satisfies the n -step MIR inequalities (4.61) and (4.62).*

Proof. Consider the point $q = (x, z^+, z^-, t) \in \text{relax}(S)$, which satisfies (4.46). By (4.46) and (4.45), we have

$$\sum_{j \in J} (a_j - 2\mu_{\alpha,b}^n(a_j)) x_j - b + 2\mu_{\alpha,b}^n(b) \leq t + z^+ + z^-. \quad (4.63)$$

Since $q \in \text{relax}(S)$, it also satisfies (4.59) and (4.60). Adding (4.59) to (4.63), we get inequality (4.61). Multiplying (4.60) by -1 and adding it to (4.63), we get inequality (4.62). This completes the proof. \square

Next we show that the above domination is strict. We do this by proving the following theorem:

Theorem 53. *Consider a point $q \in \text{relax}(S)$ and assume q does not satisfy (4.59) and (4.60) at equality. If q satisfies the n -step MIR inequality (4.61) or (4.62) at equality, then it violates the n -step conic MIR inequality (4.46).*

Proof. Let $q = (x, z^+, z^-, t)$. By the assumption, we have

$$\sum_{j \in J} a_j x_j + z^+ - z^- - t < b, \quad (4.64)$$

$$\sum_{j \in J} a_j x_j + z^+ - z^- + t > b. \quad (4.65)$$

Now if q satisfies inequality (4.61) at equality, we have

$$\sum_{j \in J} (a_j - \mu_{\alpha,b}^n(a_j)) x_j - z^- - t = b - \mu_{\alpha,b}^n(b). \quad (4.66)$$

Multiplying (4.64) by -1 and (4.66) by 2 and adding together, we get

$$\sum_{j \in J} (a_j - 2\mu_{\alpha,b}^n(a_j)) x_j - b + 2\mu_{\alpha,b}^n(b) > t + z^+ + z^-, \quad (4.67)$$

which means q violates (4.46). If q satisfies inequality (4.62) at equality, we have

$$\sum_{j \in J} \mu_{\alpha,b}^n(a_j) x_j + z^+ + t = \mu_{\alpha,b}^n(b). \quad (4.68)$$

Multiplying (4.65) by -1 and (4.68) by 2 and adding together, we get (4.67) again so q violates (4.46) in this case too. \square

Theorem 53 implies that, if the n -step MIR inequality (4.61) (or (4.62)) cuts any point in $\text{relax}(S)$, the n -step conic MIR inequality (4.46) cuts any point in $\text{relax}(S)$ that is *on* the defining hyperplane of the n -step MIR inequality (4.61) (or (4.62)) or is cut by it. Therefore, if the n -step MIR inequalities (4.61) and (4.62) cut any point in $\text{relax}(S)$ (and hence is of value), the n -step conic MIR inequality (4.46) strictly dominates them. We close this section by showing that the n -step MIR inequality for the set Y can be derived using the n -step conic MIR:

Theorem 54. *The n -step MIR inequality (2.11) for the set Y is a n -step conic MIR inequality.*

Proof. Consider the two inequalities $\sum_{j \in J} a_j x_j + s \geq b$ and $s \geq 0$ valid for the set Y . Based on (2.37), we can write these two inequalities as

$$\left| \sum_{j \in J} a_j x_j - b \right| \leq \sum_{j \in J} a_j x_j + 2s - b. \quad (4.69)$$

Now writing the n -step conic MIR inequality (4.46) for (4.69) and replacing for $\phi_{\alpha,b}^n$ from (4.45), we get

$$\sum_{j \in J} (a_j - 2\mu_{\alpha,b}^n(a_j)) x_j - (b - 2\mu_{\alpha,b}^n(b)) \leq \sum_{j \in J} a_j x_j + 2s - b, \quad (4.70)$$

which simplifies to the n -step MIR inequality (2.11). \square

IV.6 Concluding Remarks

We presented and studied new families of valid inequalities, called n -step conic MIR inequalities, for the polyhedral second-order conic sets of the form S , which have multiple integer variables. These sets not only arise in the polyhedral reformulation of SOCMIP presented in [19], but also can be used to represent any pair of mixed integer constraints according to (2.37). In that sense, the n -step conic MIR inequalities, in addition to being cutting planes for the polyhedral reformulation of the SOCMIP, can generate two-constraint cuts for linear MIP. The results in this chapter generalize the n -step MIR inequalities of Kianfar and Fathi [78] as well as the simple conic MIR inequalities of Atamtürk and Narayanan [19] and presents a unified framework that includes these inequalities as special cases. The strong facet-defining properties of the n -step conic MIR inequalities suggests that they can be effective as cutting planes. An appealing feature of these cuts is that they can be easily generated by applying

the closed-form n -step conic MIR function on a base inequality that is obtained by constraint aggregation routines like those suggested in [90].

CHAPTER V

A POLYHEDRAL STUDY OF TRIPLET FORMULATION FOR SINGLE ROW
FACILITY LAYOUT PROBLEM

In Single Row Facility Layout Problem (SRFLP), the goal is to arrange n departments on a straight line. We are given the following data: an $n \times n$ symmetric matrix $C = [c_{ij}]$, where c_{ij} denotes the average daily traffic between two departments i and j , and the length l_i of each department $i \in N = \{1, \dots, n\}$. The distance z_{ij} between two departments is considered to be the distance between their centroids. The objective is to find the permutation π that minimizes the total communication cost, i.e.

$$\min_{\pi} \sum_{i=1}^{n-1} \sum_{j=i+1}^n c_{ij} z_{ij}^{\pi}.$$

The SRFLP has several applications involving arranging rooms on a corridor, machines in a manufacturing system, and books on a shelf [72, 103, 108]. The *Minimum Linear Arrangement Problem (MLAP)* was proven to be NP-hard in [60]. The SRFLP is a generalization of MLAP and so is also NP-hard. Numerous heuristic solution approaches have been proposed for SRFLP (e.g. see [72, 80, 96, 115]).

Several exact solution techniques have also been proposed including branch and bound algorithms [108], dynamic programming [77, 103], non-linear programming [73], linear mixed integer programming [7, 8, 86]. Anjos et al. [12] and Anjos and Vanelli [13] provided lower bounds on the optimal cost of SRFLP using Semidefinite programming (SDP) relaxations. Anjos and Yen [14] computed near optimal solutions for instances

Reprinted with permission from “A polyhedral study of triplet formulation for single row facility layout problem” by Sanjeevi. S., Kianfar. K., 2012. *Discrete Applied Mathematics*, 158(16), 1861-1867, Copyright [2010] by Elsevier B.V.

with up to 100 facilities using a new SDP relaxation. Amaral and Letchford [10] conducted a polyhedral study on the distance polytope formulation of SRFLP and developed several classes of valid inequalities. They achieved quick bounds for SRFLP using LP relaxations based on these valid inequalities. They are comparable to the bounds achieved in [12].

Amaral [9] presented an alternate formulation of the SRFLP, herein referred to as the *triplet formulation*, and introduced a set of valid inequalities for it. It is shown in [9] that the linear program solved over these valid inequalities yields the optimal solution for several classical SRFLP instances of sizes $n = 5$ to $n = 30$. These problem instances are from [7, 8, 72, 73, 86, 108]. The results in [9] are comparable to the results of [13] which are based on SDP relaxation with cutting planes added.

The fact that the LP relaxation over the valid inequalities of [9] gives the optimal solution to so many instances suggests that these valid inequalities are quite strong. In this chapter we conduct a polyhedral study of the *triplet polytope*, i.e. the convex hull of feasible integer points for the triplet formulation. We prove that almost all valid inequalities introduced in [9] are indeed facet-defining for the triplet polytope. More specifically, we first show that the three polytopes (triplet polytope and its two projections defined in [9]) are of dimension $n(n-1)(n-2)/3$. After establishing the dimension of these polytopes, we then prove the aforementioned facet-defining properties.

The chapter is organized as follows: Section V.1 briefly reviews the triplet polytope, its projections, and the valid inequalities developed for them in [9]. In section V.2 we prove that these polytopes are of dimension $n(n-1)(n-2)/3$. In Section V.3 we prove the facet-defining properties of valid inequalities of [9], and we conclude in Section V.4 with a few remarks.

V.1 Triplet polytope, its projections and valid inequalities

In the triplet formulation for the SRFLP [9], a binary vector $\zeta \in \{0, 1\}^{n(n-1)(n-2)}$ is used to represent a permutation of the departments in N . Each element of ζ is identified by a triplet subscript ijk , where $i, j, k \in N$ are distinct, and

$$\zeta_{ijk} = \begin{cases} 1 & \text{if department } k \text{ lies between departments } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$$

Throughout the chapter, all the department indices used in the subscript of a single variable, coefficient, or set are assumed to be distinct and we refrain from writing this in each case. We define

$$P = \{\zeta \in \{0, 1\}^{n(n-1)(n-2)} : \zeta \text{ represents a permutation of } 1, \dots, n\},$$

and refer to the convex hull of P , i.e. $\text{conv}(P)$, as the *triplet polytope*. Based on this formulation the objective function of SRFLP can be written as

$$\min \sum_{i=1}^{n-1} \sum_{j=i+1}^n c_{ij} \left(\frac{1}{2}(l_i + l_j) + \sum_{k \neq i, k \neq j}^n l_k \zeta_{ijk} \right).$$

In [9] the following valid inequalities are presented for P :

$$0 \leq \zeta_{ijk} \leq 1 \quad i, j, k \in N \quad (5.1)$$

$$\zeta_{ijk} + \zeta_{ikj} + \zeta_{jki} = 1 \quad i, j, k \in N \quad (5.2)$$

$$\zeta_{ijd} + \zeta_{jkd} - \zeta_{ikd} \geq 0 \quad i, j, k, d \in N \quad (5.3)$$

$$\zeta_{ijd} + \zeta_{jkd} + \zeta_{ikd} \leq 2 \quad i, j, k, d \in N \quad (5.4)$$

Two projections of P are also introduced in [9]. We briefly review them here. It is clear that for any $\zeta \in P$

$$\zeta_{ijk} = \zeta_{jik} \quad 1 \leq i < j \leq n. \quad (5.5)$$

Using this identity, P can be projected onto the space $\{0, 1\}^{n'}$, where $n' = \frac{n(n-1)(n-2)}{2}$. We refer to this projection as P^1 . The projection of a vector $\zeta \in P$ will be a vector $\lambda \in P^1 \subseteq \{0, 1\}^{n'}$ with elements λ_{ijk} such that $\lambda_{ijk} = \zeta_{ijk}$ for $i, j, k \in N, i < j$. So the valid inequalities (5.1) to (5.4) can also be projected yielding the following inequalities for P^1 . Observe that (5.8), (5.9), and (5.10) are obtained from projection of (5.3).

$$0 \leq \lambda_{ijk} \leq 1 \quad i, j, k \in N, i < j \quad (5.6)$$

$$\lambda_{ijk} + \lambda_{ikj} + \lambda_{jki} = 1 \quad i, j, k \in N, i < j < k \quad (5.7)$$

$$-\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} \geq 0 \quad i, j, k, d \in N, i < j < k \quad (5.8)$$

$$\lambda_{ijd} + \lambda_{jkd} - \lambda_{ikd} \geq 0 \quad i, j, k, d \in N, i < j < k \quad (5.9)$$

$$\lambda_{ijd} - \lambda_{jkd} + \lambda_{ikd} \geq 0 \quad i, j, k, d \in N, i < j < k \quad (5.10)$$

$$\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} \leq 2 \quad i, j, k, d \in N, i < j < k \quad (5.11)$$

Amaral [9] also introduces a more complicated set of valid inequalities for $\text{conv}(P^1)$ as follows: For a positive even integer $\beta \leq n$, consider the set of distinct indices $S = \{i_t : t = 1, \dots, \beta\} \subseteq \{1, \dots, n\}$ and $d \in S$. Let (S_1, S_2) be a partition of $S \setminus \{d\}$ such that $|S_1| = \beta/2$. Then the inequality

$$\sum_{p, q \in S_1: p < q} \lambda_{pqd} + \sum_{p, q \in S_2: p < q} \lambda_{pqd} \leq \sum_{p \in S_h, q \in S_{\{1, 2\} \setminus h}: h=1, 2, p < q} \lambda_{pqd} \quad (5.12)$$

is valid for $\text{conv}(P^1)$ [9]. Inequalities (5.8), (5.9), and (5.10) are special cases of (5.12) for $\beta = 4$, as noted in [9].

P^1 can be further projected on a lower dimensional space using identity (5.7). Observe that based on (5.7) we have

$$\lambda_{ijk} = 1 - \lambda_{ikj} - \lambda_{jki} \quad i, j, k \in N, i < j < k.$$

Therefore the number of variables can be reduced to $n'' = n' - \binom{n}{3} = n(n-1)(n-2)/3$.

We refer to this projection as P^2 . The projection of a vector $\lambda \in P^1$ will be a vector $\mu \in P^2 \subseteq \{0, 1\}^{n''}$ with elements μ_{ijk} such that $\mu_{ijk} = \lambda_{ijk}$ for $i, j, k \in N, i < j, k < j$. The set of valid inequalities (5.6) to (5.12) can also be projected yielding valid inequalities for P^2 .

V.2 Dimension of convex hulls of P , P^1 and P^2

In this section, we prove that $\text{conv}(P^1)$ is of dimension n'' . Based on the projection relationships between P , P^1 and P^2 , we will then easily argue that the dimensions of $\text{conv}(P)$ and $\text{conv}(P^2)$ are n'' too. To prove that the dimension of P^1 is n'' , we will show that any hyperplane passing through all points in P^1 can be expressed as a linear combination of the set of linearly independent equalities (5.7). Since the number of these inequalities is $\binom{n}{3}$, we will have $\dim(\text{conv}(P^1)) = n' - \binom{n}{3} = n''$.

We first define some notations that we will use throughout the chapter. For any $N' \subseteq N$, we define $\Pi_{N'}$ as the set of all permutations of the departments in N' . Each $\lambda \in P^1$ corresponds to a member of Π_N . To denote the λ vector which corresponds to a given permutation $\pi \in \Pi_N$, we write λ^π . Similarly if for example $\pi^1 \in \Pi_{N \setminus \{x, y\}}$, then $\lambda^{x\pi^1 y}$ is the vector in P^1 corresponding to the permutation $x\pi^1 y$, i.e. the permutation in which x is the first department, y is the last one, and the rest are in the middle in the order π^1 . Similar notations are also used for $\zeta \in P$ and $\mu \in P^2$ that correspond to a given permutation.

Based on the definition of P^1 , λ_{ijk} is only defined when $i < j$. Therefore for any given three distinct departments i , j , and k , the variable representing whether k is between i and j or not, is λ_{ijk} if $i < j$, and is λ_{jik} if $i > j$. In many cases, just for the sake of notation simplicity, we would like to avoid differentiating between these two cases. In order to do so, wherever we have λ_{ijk} , where $i > j$, we mean λ_{jik} . We

emphasize that this is just a notational substitute and does not mean that when $i > j$ the variable λ_{ijk} really exists. We also employ this practice for a_{ijk} , the coefficient of λ_{ijk} in a hyperplane; so the reader should be careful that when $i > j$, a_{ijk} is only a notational substitute for the real coefficient a_{jik} .

The following lemma is fundamental to the result in this section.

Lemma 55. *For some given departments $x, y, z \in N$ and permutations $\pi^1 \in \Pi_{N \setminus \{x, y\}}$, $\pi^2 \in \Pi_{N \setminus \{x, y, z\}}$, if $\lambda^{xy\pi^1}$, $\lambda^{yx\pi^1}$, $\lambda^{zxy\pi^2}$, and $\lambda^{zyx\pi^2}$ lie on the hyperplane*

$$\sum_{i, j, k \in N: i < j} a_{ijk} \lambda_{ijk} = b, \quad (5.13)$$

then $a_{yxz} = a_{xzy}$.

Proof. We substitute $\lambda^{xy\pi^1}$ and $\lambda^{yx\pi^1}$ in (5.13). The left-hand sides are both equal to b , therefore

$$\sum_{i, j, k \in N: i < j} a_{ijk} \lambda_{ijk}^{xy\pi^1} = \sum_{i, j, k \in N: i < j} a_{ijk} \lambda_{ijk}^{yx\pi^1}. \quad (5.14)$$

Now observe that $\lambda_{ijk}^{xy\pi^1} = \lambda_{ijk}^{yx\pi^1}$ for any i, j, k such that $\{x, y\} \not\subset \{i, j, k\}$. Therefore a_{ijk} 's for such terms cancel out from both sides. Also $\lambda_{xyh}^{xy\pi^1} = 0$, $\lambda_{yhx}^{yx\pi^1} = 0$, $\lambda_{xhy}^{xy\pi^1} = 1$, $\lambda_{xyh}^{yx\pi^1} = 0$, $\lambda_{xhy}^{yx\pi^1} = 0$, and $\lambda_{yhx}^{yx\pi^1} = 1$ for all $h \neq x, y$. Therefore (5.14) reduces to

$$\sum_{h \neq x, y} a_{xhy} = \sum_{h \neq x, y} a_{yhx}. \quad (5.15)$$

Next we substitute the other two vectors $\lambda^{zxy\pi^2}$ and $\lambda^{zyx\pi^2}$ in (5.13) and equate the left-hand sides, we get

$$\sum_{i, j, k \in N: i < j} a_{ijk} \lambda_{ijk}^{zxy\pi^2} = \sum_{i, j, k \in N: i < j} a_{ijk} \lambda_{ijk}^{zyx\pi^2}. \quad (5.16)$$

Like above by substituting the variable values and canceling the common terms, it is

easy to see that (5.16) reduces to

$$a_{yzx} + \sum_{h \neq x, y, z} a_{xhy} = a_{xzy} + \sum_{h \neq x, y, z} a_{yhx}. \quad (5.17)$$

Subtracting (5.17) from (5.15), we get $a_{xzy} - a_{yzx} = a_{yzx} - a_{xzy}$ or $a_{yzx} = a_{xzy}$, which concludes the proof. \square

Amaral and Letchford [10] use a similar pairwise switching of departments to obtain the dimension of the distance polytope formulation they presented for SRFLP.

Theorem 56. *$\text{conv}(P^1)$ is of dimension n'' .*

Proof. $\text{conv}(P^1) \subset \mathbb{R}^{n'}$ and any $\lambda \in P^1$ satisfies the set of $\binom{n}{3}$ equalities (5.7). Clearly these set of equalities are linearly independent because no variable appears in more than one equality. Hence, $\dim(\text{conv}(P^1)) \leq n' - \binom{n}{3} = n''$. To prove that the dimension is actually equal to n'' , we just need to show that any other hyperplane like

$$\sum_{i, j, k \in N: i < j} a_{ijk} \lambda_{ijk} = b \quad (5.18)$$

satisfied by all $\lambda \in P^1$ will be a linear combination of the equalities (5.7). For this purpose observe that $\lambda^\pi \in P^1$ for any permutation $\pi \in \Pi_N$. Therefore for any three distinct departments x, y, z , by choosing any two arbitrary permutations $\pi^1 \in \Pi_{N \setminus \{x, y\}}$ and $\pi^2 \in \Pi_{N \setminus \{x, y, z\}}$, the vectors $\lambda^{xy\pi^1}, \lambda^{yx\pi^1}, \lambda^{zxy\pi^2}, \lambda^{zyx\pi^2}$ are in P^1 and so lie on (5.18). Hence by Lemma 55, $a_{yzx} = a_{xzy}$. Also for any arbitrary $\pi^3 \in \Pi_{N \setminus \{y, z\}}$, the vectors $\lambda^{yz\pi^3}, \lambda^{zy\pi^3}, \lambda^{xyz\pi^2}, \lambda^{xzy\pi^2}$ are in P^1 and so lie on (5.18). Hence again by Lemma 55, $a_{xyz} = a_{xzy}$ (note that based on our notation the order of the first two departments in the subscript does not matter). Therefore in (5.18), for any three departments x, y, z we have

$$a_{xyz} = a_{xzy} = a_{yzx}. \quad (5.19)$$

Identity (5.19) along with equalities (5.7) shows that $b = \sum_{i,j,k \in N: i < j} a_{ijk}$ and (5.18) has to be a linear combination of equalities (5.7), which concludes the proof. \square

Remember that P^1 is a projection P based on identities (5.5) and P^2 is a projection of P^1 based on identities (5.7). Therefore dimensions of $\text{conv}(P)$ and $\text{conv}(P^2)$ can be derived as a corollary to Theorem 56. This corollary is based on the following simple Lemma, which we state first.

Lemma 57. *Let A be a $n_1 \times n_2$ matrix and b be a constant n_2 -vector. If $S \subseteq \mathbb{R}^{n_1}$ and $T = \{(x, xA - b) \in \mathbb{R}^{n_1+n_2} : x \in S\}$, then $\dim(S) = \dim(T)$.*

Proof. The proof is the direct result of the fact that $x_1, \dots, x_m \in S$ are affinely independent if and only if $(x_1, x_1A - b), \dots, (x_m, x_mA - b) \in T$ are affinely independent. \square

Observe that in Lemma 57, if we denote the elements of T by (x, y) , then S is in fact the projection of T over \mathbb{R}^{n_1} based on identity $y = xA - b$.

Corollary 58. *$\text{conv}(P)$ and $\text{conv}(P^2)$ are also of dimension n'' .*

Proof. Based on the identities (5.5), $\text{conv}(P^1)$ and $\text{conv}(P)$ play the roles of S and T in Lemma 57, respectively (we would have $n_1 = n'$ and $n_1 + n_2 = 2n'$), so according to Lemma 57, $\dim(\text{conv}(P)) = \dim(\text{conv}(P^1)) = n''$.

Similarly, based on identities (5.7), $\text{conv}(P^2)$ and $\text{conv}(P^1)$ play the roles of S and T in Lemma 57, respectively (we would have $n_1 = n''$ and $n_1 + n_2 = n'$), so according to Lemma 57, $\dim(\text{conv}(P^2)) = \dim(\text{conv}(P^1)) = n''$. \square

Therefore $\text{conv}(P)$, $\text{conv}(P^1)$, and $\text{conv}(P^2)$ all have the same dimension n'' and $\text{conv}(P^2)$ is full dimensional.

V.3 Facet-defining properties of valid inequalities

In this section, we prove that inequalities (5.8), (5.9), (5.10) and (5.12) are facet-defining for $\text{conv}(P^1)$. Then as a result of Lemma 57, their corresponding inequalities for P and P^2 are also facet-defining for $\text{conv}(P)$ and $\text{conv}(P^2)$.

We note that trivial inequalities (5.6) as well as inequality (5.11) are not facet-defining in general. This can be easily seen by listing all $\lambda \in P^1$ that lie on the defining hyperplanes of these inequalities for $n = 3$ or $n = 4$ and checking their affine independence.

Theorem 59. *Inequalities (5.8), (5.9) and (5.10) are facet-defining for $\text{conv}(P^1)$.*

Proof. Consider any chosen four departments i, j, k, d ($i < j < k$). We prove the theorem for inequality (5.8). The proof for inequalities (5.9) and (5.10) is similar. By Theorem 56, we know $\dim(\text{conv}(P^1)) = n''$. Let P' be the face of $\text{conv}(P^1)$ defined by (5.8). Therefore, for every point in P' , (5.8) is satisfied at equality, i.e.

$$-\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} = 0. \quad (5.20)$$

To prove P' is a facet, we must show $\dim(P') = n'' - 1$. To show this we prove any hyperplane like

$$\sum_{e,f,g \in N: e < f} a_{efg} \lambda_{efg} = b \quad (5.21)$$

that passes through P' has to be a linear combination of the hyperplanes (5.7) and the hyperplane (5.20). First we prove the following identity:

$$a_{efg} = a_{egf} = a_{fge} \quad \text{for any } \{e, f, g\} \neq \{i, j, d\}, \{i, k, d\}, \{j, k, d\}. \quad (5.22)$$

To show this observe that the following three cases are possible:

(i). $d \notin \{e, f, g\}$: Note that any for any $\pi \in \Pi_{N \setminus \{d\}}$, $\lambda^{\pi d}$ satisfies (5.20) and hence

belongs to P' . Thus it must satisfy (5.21). So in particular, for any arbitrary $\pi^1 \in \Pi_{N \setminus \{e,f,d\}}, \pi^2 \in \Pi_{N \setminus \{e,f,g,d\}}$, the vectors $\lambda^{ef\pi^1 d}, \lambda^{fe\pi^1 d}, \lambda^{gef\pi^2 d}$, and $\lambda^{gfe\pi^2 d}$ satisfy (5.21). Therefore by Lemma 55, $a_{egf} = a_{fge}$. For the same reason, for any arbitrary $\pi^3 \in \Pi_{N \setminus \{f,g,d\}}, \lambda^{fg\pi^3 d}, \lambda^{gf\pi^3 d}, \lambda^{efg\pi^2 d}$, and $\lambda^{egf\pi^2 d}$ satisfy (5.21). So again by Lemma 55, $a_{efg} = a_{egf}$. Therefore (5.22) is true in this case.

(ii). $d \in \{e, f, g\}$ and $\{e, f, g\} \cap \{i, j, k\} = \emptyset, \{i\}$ or $\{j\}$: We assume $e = d$ (the arguments for the cases $f = d$ or $g = d$ are similar by symmetry). Now observe that for any arbitrary $\pi^1 \in \Pi_{N \setminus \{d,f\}}, \pi^2 \in \Pi_{N \setminus \{d,f,g\}}$, the vectors $\lambda^{df\pi^1}, \lambda^{fd\pi^1}, \lambda^{gdf\pi^2}$, and $\lambda^{fgd\pi^2}$ satisfy (5.20) and hence belongs to P' so they must satisfy (5.21) too. Therefore by Lemma 55, $a_{dgf} = a_{fgd}$. Also for the same reason, for any arbitrary $\pi^3 \in \Pi_{N \setminus \{d,g\}}, \lambda^{dg\pi^3}, \lambda^{gd\pi^3}, \lambda^{fdg\pi^2}$, and $\lambda^{fgd\pi^2}$ satisfy (5.21). So again by Lemma 55, $a_{dfg} = a_{fgd}$. Therefore, since $d = e$, identity (5.22) is true in this case too.

(iii). $d \in \{e, f, g\}$ and $\{e, f, g\} \cap \{i, j, k\} = \{k\}$: We assume $e = d$ and $f = k$ (the arguments for other possibilities are similar by symmetry). First observe that for any arbitrary $\pi^1 \in \Pi_{N \setminus \{g,k\}}$ and $\pi^2 \in \Pi_{N \setminus \{d,g,k\}}$, the vectors $\lambda^{gk\pi^1}, \lambda^{kg\pi^1}, \lambda^{dkg\pi^2}$, and $\lambda^{dkg\pi^2}$ satisfy (5.20) and hence belong to P' . So they satisfy (5.21). Therefore again by Lemma 55, $a_{dkg} = a_{dkg}$. Now to prove $a_{dkg} = a_{gkd}$ we cannot simply use Lemma 55 as before. The proof is as follows: Note that for any arbitrary $\pi^3 \in \Pi_{N \setminus \{d,g\}}$, the vectors $\lambda^{dg\pi^3}$ and $\lambda^{gd\pi^3}$ satisfy (5.20) so they must satisfy (5.21) too. Similar to the proof of Lemma 55, by substituting these two vectors into the left-hand side of (5.21) and equating them we get

$$\sum_{h \neq d,g} a_{dhg} = \sum_{h \neq d,g} a_{ghd}. \quad (5.23)$$

Moreover for any arbitrary $\pi^4 \in \Pi_{N \setminus \{d,g,i,k\}}$, the vectors $\lambda^{ikdg\pi^4}$ and $\lambda^{ikgd\pi^4}$ must

satisfy (5.21) for the same reason. By substitution the two vectors in the left-hand side of (5.21) and equating, we get

$$a_{gkd} + a_{gid} + \sum_{h \neq d, g, i, k} a_{dhg} = a_{dkg} + a_{dig} + \sum_{h \neq d, g, i, k} a_{ghd}. \quad (5.24)$$

Subtracting (5.24) from (5.23) we get

$$a_{dkg} - a_{gkd} + a_{dig} - a_{gid} = a_{gkd} - a_{dkg} + a_{gid} - a_{dig}. \quad (5.25)$$

But $a_{gid} = a_{dig}$ according to case (ii). So (5.25) reduces to $a_{dkg} = a_{gkd}$. Therefore identity (5.22) is true in this case too.

Moreover, for any arbitrary $\pi^1 \in \Pi_{N \setminus \{i, j, d\}}$, the vectors $\lambda^{ij\pi^1 d}, \lambda^{ji\pi^1 d}, \lambda^{dij\pi^1}, \lambda^{dji\pi^1}$ are in P' and hence satisfy (5.21). Therefore by Lemma 55,

$$a_{idj} = a_{jdi}. \quad (5.26)$$

By a similar argument, we also have

$$a_{idk} = a_{kdi}, \quad (5.27)$$

$$a_{jdk} = a_{kdj}. \quad (5.28)$$

Now observe that identities (5.22) imply that hyperplane (5.21) is a linear combination of equalities (5.7) for $\{e, f, g\} \neq \{i, j, d\}, \{i, k, d\}, \{j, k, d\}$ as well as equality (5.29) below (the coefficient of any particular equality (5.7) like $\lambda_{efg} + \lambda_{egf} + \lambda_{gfe} = 1$ in this linear combination is a_{efg} ($= a_{egf} = a_{gfe}$) and we have $b_1 = b - \sum_{\{e, f, g: e < f\} \neq \{i, j, d\}, \{i, k, d\}, \{j, k, d\}} a_{efg}$).

$$a_{ijd}\lambda_{ij d} + a_{idj}\lambda_{id j} + a_{jdi}\lambda_{jd i} + a_{ikd}\lambda_{ik d} + a_{idk}\lambda_{id k} + a_{kdi}\lambda_{kd i} + a_{jkd}\lambda_{jk d} + a_{jdk}\lambda_{jd k} + a_{kdj}\lambda_{kd j} = b_1 \quad (5.29)$$

Furthermore having identities (5.26), (5.27) and (5.28), equality (5.29) can be written

as a linear combination of equalities (5.7) for $\{i, j, d\}$, $\{i, k, d\}$, and $\{j, k, d\}$ (with coefficients a_{idj} , a_{idk} , and a_{jdk} , respectively) as well as the equality

$$(a_{ijd} - a_{idj})\lambda_{ijd} + (a_{ikd} - a_{idk})\lambda_{ikd} + (a_{jkd} - a_{jdk})\lambda_{jkd} = b_2, \quad (5.30)$$

where $b_2 = b_1 - a_{idj} - a_{idk} - a_{jdk}$. This means any point in P' must satisfy (5.30) (because it satisfies (5.21) and equalities (5.7)). In particular for any arbitrary $\pi^1 \in \Pi_{N \setminus \{d, i\}}$, the vector $\lambda^{i\pi^1 d}$ is in P' and hence satisfies (5.30). If we substitute it in (5.30), we get $b_2 = 0$. The vector $\lambda^{id\pi^1}$ also belongs to P' . Substituting this vector in (5.30) gives

$$-(a_{ijd} - a_{idj}) = a_{ikd} - a_{idk}. \quad (5.31)$$

Also for any arbitrary $\pi^2 \in \Pi_{N \setminus \{d, i, k\}}$, the vector $\lambda^{ikd\pi^2}$ is in P' and hence satisfies (5.30). Substituting this vector in (5.30) gives

$$-(a_{ijd} - a_{idj}) = a_{jkd} - a_{jdk}. \quad (5.32)$$

Using identities (5.31) and (5.32) and the fact that $b_2 = 0$, equality (5.30) reduces to

$$(a_{ijd} - a_{idj})(-\lambda_{ijd} + \lambda_{ikd} + \lambda_{jkd}) = 0. \quad (5.33)$$

Therefore, (5.33) is equality (5.20) multiplied by $a_{ijd} - a_{idj}$. So we have shown that (5.21) is a linear combination of (5.20) and the hyperplanes (5.7). This concludes the proof. \square

We mentioned that inequality (5.12) is a generalization of inequalities (5.8), (5.9), or (5.10). It turns out that this generalized inequality is also facet-defining. We prove this in Theorem 61 below; but first we prove the following lemma about a property of permutations that satisfy (5.12) at equality as we need it in proving Theorem 61.

Lemma 60. *Consider inequality (5.12) for given β , S , S_1 , S_2 and d . Let $\pi \in \Pi_N$,*

and γ_1 and γ_2 be the number of departments in S_1 and S_2 which are to the left of d in π , respectively. Then $\lambda^\pi \in P^1$ satisfies (5.12) at equality if and only if $\gamma_1 - \gamma_2 = 0$ or 1.

Proof. Let $|S_1| = \alpha$. Hence, $|S_2| = \alpha - 1$. The number of departments in S_1 and S_2 to the left of d in π is γ_1 and γ_2 , respectively; therefore, the number of departments in S_1 and S_2 to the right of d is $\alpha - \gamma_1$ and $\alpha - 1 - \gamma_2$, respectively. Now it is easy to see that in the left-hand side of (5.12), the first summation is equal to $\gamma_1(\alpha - \gamma_1)$ and the second summation is equal to $\gamma_2(\alpha - 1 - \gamma_2)$. Also the summation in the right-hand side of (5.12) is equal to $\gamma_1(\alpha - 1 - \gamma_2) + \gamma_2(\alpha - \gamma_1)$. So the validity of (5.12) is equivalent to the validity of

$$\gamma_1(\alpha - \gamma_1) + \gamma_2(\alpha - 1 - \gamma_2) \leq \gamma_1(\alpha - 1 - \gamma_2) + \gamma_2(\alpha - \gamma_1).$$

This of course reduces to

$$(\gamma_1 - \gamma_2) \leq (\gamma_1 - \gamma_2)^2, \quad (5.34)$$

which is trivial (and hence proves the validity of (5.12)). Now see that (5.34) is satisfied at equality if and only if $\gamma_1 - \gamma_2 = 0$ or 1, which means λ^π satisfies (5.12) at equality if and only if $\gamma_1 - \gamma_2 = 0$ or 1. \square

Theorem 61. *Any of inequalities (5.12) is facet-defining for $\text{conv}(P^1)$.*

Proof. Consider inequality (5.12) for given β , S , S_1 , S_2 and d . This proof is a generalization of the proof of Theorem 59 (in fact we had $S_1 = \{i, j\}$ and $S_2 = \{k\}$ in Theorem 59). Let P' be the face of $\text{conv}(P^1)$ defined by (5.12). Therefore, for every point in P' , (5.12) is satisfied at equality, i.e.

$$\sum_{p,q \in S_1: p < q} \lambda_{pqd} + \sum_{p,q \in S_2: p < q} \lambda_{pqd} - \sum_{p \in S_1, q \in S_2} \lambda_{pqd} = 0. \quad (5.35)$$

Similar to Theorem 59, we need to show that any hyperplane like

$$\sum_{e,f,g \in N: e < f} a_{efg} \lambda_{efg} = b \quad (5.36)$$

that passes through P' is a linear combination of hyperplanes (5.7) and hyperplane (5.35). First notice that as a generalization of (5.22) we prove the following identity

$$a_{efg} = a_{egf} = a_{fge} \quad \text{for any } e, f, g \text{ such that } d \notin \{e, f, g\} \text{ or } \{e, f, g\} \not\subset S. \quad (5.37)$$

To prove this see that the following cases are possible: (i). $d \notin \{e, f, g\}$; (ii). $d \in \{e, f, g\}$ and $(\{e, f, g\} \setminus d) \cap S = \emptyset$ or $\{i\}$, where $i \in S_1$; (iii). $d \in \{e, f, g\}$ and $(\{e, f, g\} \setminus d) \cap S = \{k\}$, where $k \in S_2$. The arguments for these three cases are very similar to the arguments for cases (i), (ii), and (iii) in the proof of Theorem 59, respectively. The λ vectors used are exactly the same and the reason why they satisfy (5.35) is Lemma 60 because in all given permutations $\gamma_1 - \gamma_2 = 0$ or 1. In case (iii), the i that is used in the proof of Theorem 59 represents any arbitrary member of S_1 .

Moreover for any $p, q \in S_1$ and any arbitrary permutation $\pi^1 \in \Pi_{N \setminus \{p, q, d\}}$, the vectors $\lambda^{pq\pi^1 d}$, $\lambda^{qp\pi^1 d}$, $\lambda^{dpq\pi^1}$, and $\lambda^{dqp\pi^1}$ satisfy (5.35) by Lemma 60, so they must satisfy (5.36). Therefore by Lemma 55,

$$a_{pdq} = a_{qdp} \quad \text{for all } p, q \in S_1. \quad (5.38)$$

By a similar argument, we also have

$$a_{sdt} = a_{tds} \quad \text{for all } s, t \in S_2, \quad (5.39)$$

$$a_{pds} = a_{sdp} \quad \text{for all } p \in S_1, s \in S_2. \quad (5.40)$$

Now observe that having identities (5.38), (5.39) and (5.40), hyperplane (5.36) can

be written as a linear combination of equalities (5.7) as well as the equality

$$\sum_{p,q \in S_1, p < q} (a_{pqd} - a_{pdq}) \lambda_{pqd} + \sum_{s,t \in S_2, s < t} (a_{std} - a_{sdt}) \lambda_{std} + \sum_{p \in S_1, s \in S_2} (a_{psd} - a_{pds}) \lambda_{psd} = b_1. \quad (5.41)$$

Now for any arbitrary $\pi^1 \in \Pi_{N \setminus \{d\}}$, $\lambda^{\pi^1 d}$ is in P' . Substituting this vector in (5.41) gives $b_1 = 0$. Moreover, for any $p, q \in S_1$, $s \in S_2$ and arbitrary $\pi^2 \in \Pi_{N / \{d, p, q, s\}}$, the vector $\lambda^{pqds\pi^2}$ belongs to P' . Substituting this vector in (5.41) gives

$$a_{pqd} - a_{pdq} = -(a_{psd} - a_{pds}) \text{ for all } p, q \in S_1, s \in S_2. \quad (5.42)$$

Also for any $p \in S_1$, $s, t \in S_2$ and arbitrary $\pi^3 \in \Pi_{N \setminus \{d, i, k\}}$, the vector $\lambda^{ptds\pi^3}$ is in P' . Substituting this vector in (5.41) gives

$$a_{std} - a_{sdt} = -(a_{psd} - a_{pds}) \text{ for all } p \in S_1, s, t \in S_2. \quad (5.43)$$

Identities (5.42) and (5.43) imply that all coefficients in equality (5.41) are equal. Let the constant K denote their common value. Therefore (5.41) reduces to

$$K \left(\sum_{p,q \in S_1, p < q} \lambda_{pqd} + \sum_{s,t \in S_2, s < t} \lambda_{std} - \sum_{p \in S_1, s \in S_2} \lambda_{psd} \right) = 0. \quad (5.44)$$

Therefore, (5.44) is equality (5.35) multiplied by K . So we have shown that (5.36) is a linear combination of (5.35) and the hyperplanes (5.7). This concludes the proof. \square

Corollary 62. *Inequalities (5.12), written for ζ instead of λ , and inequalities (5.3) are facet-defining for $\text{conv}(P)$. Also the projections of inequalities (5.8), (5.9), (5.10), and (5.12) for P^2 are facet-defining for $\text{conv}(P^2)$.*

Proof. The proof is a direct result of Theorems 59 and 61 and Lemma 57 applied to the faces defined by these inequalities. \square

V.4 Conclusions

We proved that the convex hulls of the triplet formulation for SRFLP and its projections [9] are of dimension $n(n-1)(n-2)/3$, where n is the number of departments. We also showed that many valid inequalities presented in [9] for this polytope are facet-defining. Our result provides a theoretical support for the fact that the LP solution over these valid inequalities gives the optimal solution for all instances studied in [9]. A possible direction for future research is to develop new classes of valid inequalities and facets for the triplet polytope.

CHAPTER VI

CONCLUSION AND FUTURE RESEARCH

In this dissertation, we introduced several classes of new valid inequalities for general and special structured linear MIPs, and general SOCMIPs, and established several theoretical properties for these inequalities.

First, we developed the type I and type II mixed n -step MIR inequalities for the n -mixing set, a generalization of the mixing set [70] with each constraint having multiple integer variables, and identified conditions under which they are facet-defining. We then used mixed n -step MIR to generate multi-row valid inequalities for general MIPs, and generalized well-known families of inequalities for the capacitated lot-sizing and facility location problems to the multi-module case. We also presented computational results showing the effectiveness of the mixed n -step MIR cuts for small MIPLIB instances and random multi-module lot sizing instances.

Next, we introduced the n -step conic MIR inequalities for PSOC mixed integer sets. We first derived the n -step conic MIR inequality for a certain PSOC set with n integer variables, and identified new valid inequalities for this set based on n -step conic MIR inequalities for lower-dimensional PSOC sets. We then proved that all of the above inequalities are facet-defining for the convex hull of this set. We also identified necessary and sufficient conditions for the PSOC form of this inequality to be valid. Then, we used the n -step conic MIR facets for PSOC sets to derive the n -step conic MIR inequality for a general PSOC set and identified facet-defining conditions for this inequality. We generated these inequality using functions called the n -step conic MIR functions. We further showed the relationship between n -step conic MIR

inequalities and n -step conic MIR inequalities, and proved that the n -step conic MIR inequality for the PSOC set dominates the n -step MIR inequality associated with the linear constraints defining the PSOC set.

Finally, we conducted a polyhedral study of the triplet formulation, a MIP formulation of the SRFLP introduced by Amaral [9]. For any number of departments n , we proved that the dimension of the triplet polytope is $n(n-1)(n-2)/3$. We then proved that several valid inequalities presented in [9] for this polytope are facet-defining. These results provide theoretical support for the fact that the linear program solved over these valid inequalities gives the optimal solution for all instances studied by Amaral [9].

The research in this dissertation opens new doors to several theoretical and computational research directions. We present a brief summary of the main research paths that begin from the results in this dissertation in the following sections.

VI.1 Theoretical Research

1. **Facets for infinite group polyhedra.** Mixed n -step MIR has an interesting relationship with the infinite group problem [65, 66, 67, 68]. We have observed evidence for the fact that the function σ used to generate mixing-based inequalities for general linear MIPs also defines valid inequalities for the infinite group problem. An interesting research question is whether these inequalities are also extreme inequalities, and whether new extreme functions can be identified for the group problem using the function σ .
2. **Valid inequalities for polyhedra with upper bounds on variables.** Atamtürk and Günlük [17] introduced the *mingling inequalities* by incorporating information about upper bounds on integer variables into the MIR inequalities. Atamtürk and Kianfar [18] developed the n -step mingling inequalities, which are

generalized mingling inequalities obtained by incorporating upper bounds on integer variables into the n -step MIR inequalities. A research path investigates whether mingling-based inequalities can also be developed by incorporating upper bound information into mixed n -step MIR and n -step conic MIR inequalities.

3. **Inequalities based on more complicated cones.** The n -step conic MIR is developed based on the facets of a mixed-integer set defined by a two-hyperplane cone (represented by the PSOC constraint). In future, we intend to study the polyhedral structure of more complicated cones defined by multiple hyperplanes, identify new facets for such cones, and use these facets to generate valid inequalities for more general mixed-integer sets.
4. **Valid inequalities for other special structure MIPs.** Another exciting research path is to study whether mixed n -step MIR can generate new valid inequalities for special structure MIPs such as multi-capacity network design. Studying the facet-defining property of inequalities based on mixed n -step MIR for multi-capacity lot sizing, facility location and network design is also of interest. We have started some preliminary research on the facet-defining property of the inequalities (3.54) and (3.55) for the multi-capacity lot sizing problem.
5. **Weaker validity conditions.** The mixed n -step MIR and n -step conic MIR inequalities introduced in this dissertation require the condition (3.1) on parameters to be satisfied. Studying the n -mixing sets $Q^{m,n}$ and PSOC sets Q^n in which these conditions are relaxed and possible resulting valid inequalities is another research direction to be explored.

VI.2 Computational Research

Our preliminary computational experiments with mixed n -step MIR inequalities for general MIPs are quite promising in terms of improvement in the integrality gap over 1-step MIR cuts. Our results for random lot-sizing instances are a clear evidence of the fact that mixed n -step MIR cuts are very efficient cutting planes for special structure MIPs. Due to the encouraging computational results, we plan to develop heuristics to add our new classes of valid inequalities efficiently in branch-and-cut algorithms, and plan to perform computational experiments for other special structure MIPs with our cuts.

REFERENCES

- [1] Aardal, K., Pochet, Y., Wolsey, L.A.: Capacitated facility location: valid inequalities and facets. *Mathematics of Operations Research* **20**(3), 562–582 (1995)
- [2] Aardal, K., Pochet, Y., Wolsey, L.A.: Erratum: Capacitated facility location: valid inequalities and facets. *Mathematics of Operations Research* **21**, 253–256 (1996)
- [3] Aardal, K., Weismantel, R., Wolsey, L.A.: Non-standard approaches to integer programming. *Discrete Applied Mathematics* **123**, 5–74 (2002)
- [4] Abhishek, K., Leyffer, S., Linderoth, J.: FilMINT: an outer approximation-based solver for convex mixed-integer nonlinear programs. *INFORMS Journal on Computing* **22**(4), 555–567 (2010)
- [5] Agra, A., Constantino, M.: Mixing polyhedra with two non divisible coefficients. *Mathematical Programming* (2011). URL <http://www.springerlink.com/content/g527545038402146/>
- [6] Alizadeh, F., Goldfarb, D.: Second-order cone programming. *Mathematical Programming* **95**, 3–51 (2001)
- [7] Amaral, A.R.S.: On the exact solution of a facility layout problem. *European Journal of Operational Research* **173**(2), 508–518 (2006)
- [8] Amaral, A.R.S.: An exact approach to the one-dimensional facility layout problem. *Operations Research* **56**(4), 1026–1033 (2008)
- [9] Amaral, A.R.S.: A new lower bound for the single row facility layout problem. *Discrete Applied Mathematics* **157**(1), 183–190 (2009)

- [10] Amaral, A.R.S., Letchford, A.N.: A polyhedral approach to the single row facility layout problem. Available on Optimization Online (2008). URL http://www.optimization-online.org/DB_HTML/2008/03/1931.html
- [11] Andersen, K., Louveaux, Q., Weismantel, R., Wolsey, L.A.: Inequalities from two rows of a simplex tableau. In: M. Fischetti, D.P. Williamson (eds.) Integer Programming and Combinatorial Optimization, 12th International IPCO Conference, pp. 1–15. Springer (2007)
- [12] Anjos, M.F., Kennings, A., Vannelli, A.: A semidefinite optimization approach for the single-row layout problem with unequal dimensions. *Discrete Optimization* **2**(2), 113–122 (2005)
- [13] Anjos, M.F., Vannelli, A.: Computing globally optimal solutions for single-row layout problems using semidefinite programming and cutting planes. *INFORMS Journal on Computing* **20**(4), 611–617 (2008)
- [14] Anjos, M.F., Yen, G.: Provably near-optimal solutions for very large single-row facility layout problems. *Optimization Method and Software* **24**(4-5), 805–817 (2009)
- [15] Atamtürk, A.: On the facets of the mixed-integer knapsack polyhedron. *Mathematical Programming* **98**, 145–175 (2003)
- [16] Atamtürk, A.: Sequence independent lifting for mixed-integer programming. *Operations Research* **52**, 487–490 (2004)
- [17] Atamtürk, A., Günlük, O.: Mingling: mixed-integer rounding with bounds. *Mathematical Programming* **123**, 315–338 (2010)

- [18] Atamtürk, A., Kianfar, K.: n-step mingling inequalities: new facets for the mixed-integer knapsack set. *Mathematical Programming* **132**, 79–98 (2012)
- [19] Atamtürk, A., Narayanan, V.: Conic mixed-integer rounding cuts. *Mathematical Programming Ser. A* **122**, 1–20 (2010)
- [20] Atamtürk, A., Narayanan, V.: Lifting for conic mixed-integer programming. *Mathematical Programming* **126**, 351–363 (2011)
- [21] Atamtürk, A., Muñoz, J.C.: A study of the lot-sizing polytope. *Mathematical Programming* **99**(3), 443–465 (2004)
- [22] Balas, E.: Intersection cuts – a new type of cutting planes for integer programming. *Operations Research* **19**, 19–39 (1971)
- [23] Balas, E.: Disjunctive programs: Cutting planes from logical conditions. In: O.L. Mangasarian et al. (ed.) *Nonlinear Programming*, vol. 2, pp. 279–312. Academic Press, New York (1975)
- [24] Balas, E., Ceria, S., Cornuéjols, G.: A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Math. Programming* **58**, 295–324 (1993)
- [25] Basu, A., Conforti, M., Cornuéjols, G., Zambelli, G.: Minimal inequalities for an infinite relaxation of integer programs. <http://www.math.unipd.it/~giacomo/> (2009)
- [26] Ben-Tal, A., Nemirovski, A.S.: *Lectures on modern convex optimization: analysis, algorithms, and engineering applications*. Society for Industrial and Applied Mathematics, Philadelphia, PA (2001)
- [27] Bertsimas, D., Shioda, R.: Algorithm for cardinality-constrained quadratic optimization. *Computational Optimization and Applications* **43**, 1–22 (2007)

- [28] Bienstock, D.: Computational study of a family of mixed-integer quadratic programming problems. *Mathematical Programming* **74**, 121–140 (1996)
- [29] Bonami, P., Biegler, L.T., Conn, A.R., Cornuéjols, G., Grossmann, I.E., Laird, C.D., Lee, J., Lodi, A., Margot, F., Sawaya, N., Wächter, A.: An algorithmic framework for convex mixed integer nonlinear programs. *Discrete Optimization* **5**, 186–204 (2008)
- [30] Borchers, B., Mitchell, J.E.: An improved branch and bound algorithm for mixed integer nonlinear programs. *Computers & Operations Research* **21**(4), 359–367 (1994)
- [31] Borozan, V., Cornuéjols, G.: Minimal valid inequalities for integer constraints. *Mathematics of Operations Research* **34**, 538–546 (2009)
- [32] Boyd, S., Vandenberghe, L.: *Convex Optimization*. Cambridge University Press, Cambridge, UK (2004)
- [33] Caprara, A., Fischetti, M.: Branch-and-Cut algorithms. In: M. Dell’Amico, F. Maffioli, S. Martello (eds.) *Annotated Bibliographies in Combinatorial Optimization*, pp. 45–63. Wiley (1997)
- [34] Caprara, A., Letchford, A.N.: On the separation of split cuts and related inequalities. *Mathematical Programming* **94**, 279–294 (2003)
- [35] Çezik, M.T., Iyengar, G.: Cuts for mixed 0-1 conic programming. *Mathematical Programming* **104**, 179–202 (2005)
- [36] Chvátal, V.: Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics* **4**(4), 305 – 337 (1973)

- [37] Conforti, M., Cornuéjols, G., Zambelli, G.: A geometric perspective on lifting. <http://www.math.unipd.it/~giacomo/> (2009)
- [38] Conforti, M., Di Summa, M., Wolsey, L.A.: The mixing set with flows. *SIAM Journal on Discrete Mathematics* **21**, 396 (2007)
- [39] Conforti, M., Di Summa, M., Wolsey, L.A.: The mixing set with divisible capacities. In: A. Lodi, A. Panconesi, G. Rinaldi (eds.) *Integer Programming and Combinatorial Optimization*, vol. 5035, pp. 435–449. Springer, Berlin (2008)
- [40] Conforti, M., Zambelli, G.: The mixing set with divisible capacities: A simple approach. *Operations Research Letters* **37**, 379–383 (2009)
- [41] Constantino, M., Miller, A.J., Van Vyve, M.: Mixing MIR inequalities with two divisible coefficients. *Mathematical Programming* **123**, 451–483 (2009)
- [42] Cook, W., Kannan, R., Schrijver, A.: Chvátal closures for mixed integer programming problems. *Mathematical Programming* **47**, 155–174 (1990)
- [43] Cornuéjols, G., Margot, F.: On the facets of mixed integer programs with two integer variables and two constraints. *Mathematical Programming*, To appear, Available Online First at <http://dx.doi.org/10.1007/s10107-008-0221-1> (2008)
- [44] Crowder, H., Johnson, E.L., Padberg, M.: Solving Large-Scale Zero-One linear programming problems. *Operations Research* **31**, 803–834 (1983)
- [45] Dash, S., Goycoolea, M., Günlük, O.: Two-Step MIR inequalities for mixed integer programs. *INFORMS Journal on Computing* **Articles in Advance**, 1–14 (2009)
- [46] Dash, S., Günlük, O.: Valid inequalities based on simple mixed-integer sets. *Mathematical Programming* **105**, 29–53 (2006)

- [47] Dash, S., Günlük, O.: On mixing inequalities: rank, closure, and cutting-plane proofs. *SIAM Journal on Optimization* **20**, 1090 (2009)
- [48] Dey, S.: A note on the split rank of intersection cuts. *Mathematical Programming* **130**, 107–124 (2011)
- [49] Dey, S., Louveaux, Q.: Split rank of triangle and quadrilateral inequalities. *Optimization Online Manuscript* (2009)
- [50] Dey, S., Richard, J.: Sequential-merge facets for two-dimensional group problems. In: M. Fischetti, D.P. Williamson (eds.) *Integer Programming and Combinatorial Optimization*, 12th International IPCO Conference, pp. 30–42. Springer (2007)
- [51] Dey, S., Richard, J.P.: Facets of two-dimensional infinite group problems. *Mathematics of Operations Research* **33**, 140–166 (2008)
- [52] Dey, S.S., Wolsey, L.A.: Composite lifting of group inequalities and an application to two-row mixing inequalities. *Discrete Optimization* **7**, 256–268 (2010)
- [53] Di Summa, M., Wolsey, L.A.: Mixing sets linked by bidirected paths. CORE Discussion Papers 2010063, Université catholique de Louvain, Center for Operations Research and Econometrics (2010)
- [54] Duran, M., Grossmann, I.: An outer-approximation algorithm for a class of mixed-integer nonlinear programs. *Mathematical Programming* **36**(3), 307–339 (1986)
- [55] Elf, M., Gutwenger, C., Jünger, M., Rinaldi, G.: Branch-and-cut algorithms for combinatorial optimization and their implementation in ABACUS. In: D. Naddef, M. Jünger (eds.) *Computational Combinatorial Optimization*, pp. 157–221. Springer, Berlin (2001)

- [56] Espinoza, D.G.: Computing with multi-row gomory cuts. In: Proceedings of the 13th international conference on Integer programming and combinatorial optimization, IPCO'08, pp. 214–224. Springer-Verlag, Berlin (2008)
- [57] Fletcher, R., Leyffer, S.: Solving mixed integer nonlinear programs by outer approximation. *Mathematical Programming* **66**(1), 327–349 (1994)
- [58] Fukasawa, R., Goycoolea, M.: On the exact separation of mixed integer knapsack cuts. *Mathematical Programming* **128**(1-2), 19–41 (2009)
- [59] Fukasawa, R., Günlük, O.: Strengthening lattice-free cuts using non-negativity. Optimization Online, http://www.optimization-online.org/DB_HTML/2009/05/2296.html (2009)
- [60] Garey, M.R., Johnson, D.S.: Computers and intractability: an introduction to the theory of NP-completeness. W.H. Freeman & Co, New York (1979)
- [61] Geoffrion, A.M.: Generalized benders decomposition. *Journal of Optimization Theory and Applications* **10**(4), 237–260 (1972)
- [62] Gomory, R.E.: Outline of an algorithm for integer solutions to linear programs. *Bulletin of the American Mathematical Society* **64**, 275–278 (1958)
- [63] Gomory, R.E.: An algorithm for the mixed integer problem. Technical Report RM-2597, The Rand Corporation, Santa Monica, CA (1960)
- [64] Gomory, R.E.: An algorithm for integer solutions to linear programs. In: R.L. Graves, P. Wolfe (eds.) *Recent Advances in Mathematical Programming*, pp. 269–302. McGraw-Hill, New York (1963)
- [65] Gomory, R.E.: Some polyhedra related to combinatorial problems. *Linear Algebra and its Applications* **2**(4), 451 – 558 (1969)

- [66] Gomory, R.E., Johnson, E.L.: Some continuous functions related to corner polyhedra. *Mathematical Programming* **3**, 23–85 (1972)
- [67] Gomory, R.E., Johnson, E.L.: Some continuous functions related to corner polyhedra, ii. *Mathematical Programming* **3**, 359–389 (1972)
- [68] Gomory, R.E., Johnson, E.L.: T-space and cutting planes. *Mathematical Programming* **96**, 341–375 (2003)
- [69] Grötschel, M., Padberg, M.W.: On the symmetric travelling salesman problem II: lifting theorems and facets. *Mathematical Programming* **16**, 281–302 (1979)
- [70] Günlük, O., Pochet, Y.: Mixing mixed-integer inequalities. *Mathematical Programming* **90**, 429–457 (2001)
- [71] Gupta, O.K., Ravindran, A.: Branch and bound experiments in convex nonlinear integer programming. *Management Science* **31**(12), 1533–1546 (1985)
- [72] Heragu, S.S., Kusiak, A.: Machine layout problem in flexible manufacturing systems. *Operations Research* **36**(2), 258–268 (1988)
- [73] Heragu, S.S., Kusiak, A.: Efficient models for the facility layout problem. *European Journal of Operational Research* **53**(1), 1–13 (1991)
- [74] Johnson, E.L.: On the group problem for mixed integer programming. In: *Approaches to Integer Programming, Mathematical Programming Studies*, vol. 2, pp. 137–179. Springer, Berlin (1974)
- [75] Johnson, E.L., Nemhauser, G.L., Savelsbergh, M.W.P.: Progress in linear programming-based algorithms for integer programming: An exposition. *INFORMS Journal on Computing* **12**, 2–23 (2000)

- [76] Jünger, M., Reinelt, G., Thienel, S.: Practical problem solving with cutting plane algorithms in combinatorial optimization. In: W. Cook, L. Lovász, P. Seymour (eds.) *Combinatorial Optimization. DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pp. 111–152. AMS (1995)
- [77] Karp, R.M., Held, M.: Finite-state processes and dynamic programming. *SIAM Journal on Applied Mathematics* **15**(3), 693–718 (1967)
- [78] Kianfar, K., Fathi, Y.: Generalized mixed integer rounding inequalities: facets for infinite group polyhedra. *Mathematical Programming* **120**, 313–346 (2009)
- [79] Kojima, M., Tunçel, L.: Cones of matrices and successive convex relaxations of nonconvex sets. *SIAM J. on Optimization* **10**, 750–778 (1999)
- [80] Kumar, K.R., Hadjinicola, G.C., Li Lin, T.: A heuristic procedure for the single-row facility layout problem. *European Journal of Operational Research* **87**(1), 65–73 (1995)
- [81] Land, A.H., Doig, A.G.: An automatic method of solving discrete programming problems. *Econometrica* **28**(3), 497–520 (1960)
- [82] Lasserre, J.B.: An explicit exact SDP relaxation for nonlinear 0-1 programs. In: *Proceedings of the 8th International IPCO Conference on Integer Programming and Combinatorial Optimization*, pp. 293–303. Springer-Verlag, London, UK (2001)
- [83] Leyffer, S.: Integrating SQP and branch-and-bound for mixed integer nonlinear programming. *Computational Optimization and Applications* **18**(3), 295–309 (2001)

- [84] Lobo, M., Vandenberghe, L., Boyd, S., Lebret, H.: Applications of second-order cone programming. *Linear Algebra Appl.* **284**, 193–228 (1998)
- [85] Lobo, M.S., Fazel, M., Boyd, S.: Portfolio optimization with linear and fixed transaction costs. *Annals of Operations Research* **152**, 341–365 (2007)
- [86] Love, R.F., Wong, J.Y.: On solving a one-dimensional space allocation problem with integer programming. *INFOR* **14**(2), 139–143 (1976)
- [87] Luo, J., Pattipati, K.R., Willett, P., Levchuk, G.M.: Fast optimal and suboptimal any-time algorithms for CDMA multiuser detection based on branch and bound. *IEEE Transactions on Communications* **52**(4), 632–642 (2004)
- [88] Luo, Z.: Applications of convex optimization in signal processing and digital communication. *Mathematical Programming, Ser. B* **97**(1-2), 177–207 (2003)
- [89] Marchand, H., Martin, A., Weismantel, R., Wolsey, L.A.: Cutting planes in integer and mixed integer programming. *Discrete Applied Mathematics* **123**, 397–446 (2002)
- [90] Marchand, H., Wolsey, L.A.: Aggregation and mixed integer rounding to solve MIPs. *Operations Research* **49**, 363–371 (2001)
- [91] Maringer, D., Kellerer, H.: Optimization of cardinality constrained portfolios with a hybrid local search algorithm. *OR Spectrum* **25**, 481–495 (2003)
- [92] Martin, A.: General mixed integer programming: computational issues for branch-and-Cut algorithms. In: D. Naddef, M. Jünger (eds.) *Computational Combinatorial Optimization*, pp. 157–221. Springer (2001)
- [93] Masihabadi, S., Sanjeevi, S., Kianfar, K.: n-step conic mixed integer rounding cuts. *Mathematical Programming* (to be submitted) (2011)

- [94] Mitchell, J.: Branch-and-cut methods for combinatorial optimization problems. In: P. Pardalos et al. (ed.) *Handbook of Applied Optimization*. Oxford University Press, Oxford, GB (2001)
- [95] Mobasher, A., Taherzadeh, M., Sotirov, R., Khandani, A.K.: A near-maximum-likelihood decoding algorithm for MIMO systems based on semi-definite programming. *IEEE Transactions on Information Theory* **53**, 3869–3886 (2007)
- [96] Neghabat, F.: An efficiency equipment-layout algorithm. *Operations Research* **22**(3), 622–628 (1974)
- [97] Nemhauser, G.L., Wolsey, L.A.: *Integer and combinatorial optimization*. Wiley-Interscience, New York, USA (1988)
- [98] Nemhauser, G.L., Wolsey, L.A.: A recursive procedure to generate all cuts for 0-1 mixed integer programs. *Mathematical Programming* **46**, 379–390 (1990)
- [99] Nesterov, Y., Nemirovskii, A.: *Interior-point polynomial algorithms in convex programming*, vol. 13. SIAM, Philadelphia, USA (1994)
- [100] Padberg, M.W.: On the facial structure of set packing polyhedra. *Mathematical Programming* **5**, 199–215 (1973)
- [101] Padberg, M.W., Rinaldi, G.: Optimization of a 532-city symmetric traveling salesman problem by branch-and-cut. *Operations Research Letters* **6**, 1–7 (1987)
- [102] Padberg, M.W., Roy, T.J.v., Wolsey, L.A.: Valid linear inequalities for fixed charge problems. *Operations Research* **33**, 842–861 (1985)
- [103] Picard, J.C., Queyranne, M.: On the one-dimensional space allocation problem. *Operations Research* **29**(2), 371–391 (1981)

- [104] Pochet, Y., Wolsey, L.A.: Lot-sizing with constant batches: formulation and valid inequalities. *Mathematics of Operations Research* **18**(4), 767–785 (1993)
- [105] Quesada, I., Grossmann, I.: An LP/NLP based branch and bound algorithm for convex MINLP optimization problems. *Computers & Chemical Engineering* **16**(10-11), 937–947 (1992)
- [106] Sanjeevi, S., Kianfar, K.: A polyhedral study of triplet formulation for single row facility layout problem. *Discrete Applied Mathematics* **158**(16), 1861 – 1867 (2010)
- [107] Sherali, H.D., Tuncbilek, C.H.: A reformulation-convexification approach for solving nonconvex quadratic programming problems. *Journal of Global Optimization* **7**, 1–31 (1995)
- [108] Simmons, D.M.: One-dimensional space allocation: an ordering algorithm. *Operations Research* **17**(5), 812–826 (1969)
- [109] Stubbs, R.A., Mehrotra, S.: A branch-and-cut method for 0-1 mixed convex programming. *Mathematical Programming* **86**, 515–532 (1999)
- [110] Stubbs, R.A., Mehrotra, S.: Generating convex polynomial inequalities for mixed 0-1 programs. *Journal of Global Optimization* **24**, 311–332 (2002)
- [111] Tawarmalani, M., Sahinidis, N.V.: Global optimization of mixed-integer nonlinear programs: A theoretical and computational study. *Mathematical Programming* **99**, 2004 (2003)
- [112] Tawarmalani, M., Sahinidis, N.V.: A polyhedral branch-and-cut approach to global optimization. *Mathematical Programming* **103**, 225–249 (2005)

- [113] Van Vyve, M.: The continuous mixing polyhedron. *Mathematics of Operations Research* **30**, 441–452 (2005)
- [114] Vielma, J.P., Ahmed, S., Nemhauser, G.L.: A lifted linear programming branch-and-bound algorithm for mixed-integer conic quadratic programs. *INFORMS Journal on Computing* **20**(3), 438–450 (2008)
- [115] Vrat, P., Solimanpur, M., Shankar, R.: An ant algorithm for the single row layout problem in flexible manufacturing systems. *Computers and Operations Research* **32**(3), 583–598 (2005)
- [116] Westerlund, T., Pettersson, F.: An extended cutting plane method for solving convex MINLP problems. *Computers & Chemical Engineering* **19, Supplement 1**, 131–136 (1995)
- [117] Wolsey, L.A.: *Integer programming*. Wiley-Interscience, New York, USA (1998)
- [118] Wolsey, L.A.: Solving multi-item lot-sizing problems with an MIP solver using classification and reformulation. *Management Science* **48**(12), 1587–1602 (2002)
- [119] Zhao, M., de Farias, I.: The mixing-MIR set with divisible capacities. *Mathematical Programming* **115**, 73–103 (2008)
- [120] Zhao, M., de Farias, I.: A note on the continuous mixing set. *Operations Research Letters* **36**, 726–733 (2008)